1. Suppose E is an extension field of F. If $\alpha \in E$ is algebraic over F and $\beta \in F(\alpha)$, prove that $\deg(\beta, F)$ divides $\deg(\alpha, F)$.
2. Suppose E is a finite extension of a field F and $[E : F]$ is a prime number. Prove that $E = F(\alpha)$ for every $\alpha \in E$ which is not already in F.
3. Prove that the extensions $\mathbb{Q}(\sqrt{5}+\sqrt{2})$ and $\mathbb{Q}(\sqrt{2}, \sqrt{10})$ of $\mathbb{Q}$ are equal.
4. Suppose I and J are ideals of a ring R. Let
$$I + J = \{a + b \mid a \in I, b \in J\}.$$
Prove that $I + J$ is an ideal of R.
5. Suppose R is a ring with unity and I is an ideal of R. Prove that if I contains a unit, then $I = R$.
6. Let $\phi\colon R \to R'$ be a surjective ring homomorphism. Prove that if S is a subring of R, then $\phi(S)$ is subring of R'.
7. Prove using the definitions that x is neither a unit nor a zero divisor in $\mathbb{Z}[x]$.
8. Let p, q be prime numbers. Find all elements of $\mathbb{Z}_p \times \mathbb{Z}_q$ which are their own multiplicative inverses (i.e., that are solutions to the equation $x^2 - 1 = 0$), and justify your answer.
9. Suppose R is a ring with unity. If R has finite characteristic n and n is not prime, prove (from definitions) that R has zero divisors.
10. Suppose R is a ring with no zero divisors, and let $a, b, c \in R$. Prove that the left cancellation law holds, i.e., that if $ab = ac$ and $a \neq 0$, then $b = c$.
11. Suppose S and T are subrings of a ring R. Prove that $S \cap T$ is a subring of R. What if S and T are ideals – is $S \cap T$ also an ideal?
12. Determine, with proof, the isomorphism type of $\mathbb{Z}_6 \times \mathbb{Z}_8/\langle(0, 2)\rangle$, using the fundamental theorem of finitely generated abelian groups.
13. Determine, with proof, the isomorphism type of $\mathbb{Z} \times \mathbb{Z}/\langle(1, 1)\rangle$, using the fundamental theorem of finitely generated abelian groups.
14. Let G be a group with $|G| > 1$ such that G has no nontrivial proper subgroups. Show that G is a finite cyclic group of prime order.

15. Show that the alternating group A_4 contains a subgroup isomorphic to the Klein-4 group.

16. Let G be a group, and let $Z(G)$ be the center of G. Show that if $G/Z(G)$ is cyclic, then G is abelian.

17. Show that if G is a finite group such that for all $g \in G$ we have $g^2 = e$, then

    (a) G is abelian,

    (b) $|G| = 2^n$ for some positive integer n,

    (c) $G \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

18. A subgroup H of a group G is called a **characteristic subgroup** of G if for all $\phi \in \mathrm{Aut}(G)$ we have $\phi(H) = H$.

    (a) Show that if H is a characteristic subgroup of G, then $H \triangleleft G$.

    (b) Let G be a group, H a normal subgroup of G, and K a characteristic subgroup of H. Show that K is a normal subgroup of G.

19. Let a be a nilpotent element in a commutative ring R with unity (that is, there exists $n \in \mathbb{Z}$ such that $a^n = 0$). Show that

    (a) $a = 0$ or a is a zero divisor.

    (b) ax is nilpotent for all $x \in R$.

    (c) $1 + a$ is a unit in R.

    (d) If u is a unit in R, then $u + a$ is also a unit in R.

20. Let $I = n\mathbb{Z}$ and $J = m\mathbb{Z}$ be two ideals in $\mathbb{Z}$. Then $I + J = k\mathbb{Z}$ for some $k \in \mathbb{Z}$. Express k in terms of n and m.

21. Let R be a commutative ring. The **annihilator** of R is defined as follows:

$$\mathrm{Ann}(R) = \{a \in R \mid ax = 0 \text{ for all } x \in R\}.$$

    Show that $\mathrm{Ann}(R)$ is an ideal of R.

22. Let R be a commutative ring and I and ideal in R. The **radical** of I is defined as

$$\mathrm{rad}(I) = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{Z}\}.$$

    Show that $\mathrm{rad}(I)$ is an ideal of R containing I.