Exercises 13: 18, 25-28, 47

Additional exercises:

1. Let $G = \mathbb{R}^3$ be the group of real 3–tuples with component-wise addition (vector addition).
   (a) What geometric object is the set
   $$H = \{(x, y, z \in \mathbb{R}^3 \mid 3(x-1) + 2(y+5) - (z+1) = 6\}?$$
   Prove that H is a subgroup of G.
   (b) The cyclic subgroup generated by $(0, 0, 0)$ is the trivial subgroup of G, but all other cyclic subgroups are more interesting. Let K be the cyclic subgroup generated by a nonindentity element (a, b, c). Give an algebraic and geometric description of K.
   (c) Let L be the line through the origin and the point $(2, 3, 5)$. Is L a subgroup of G? If so, is it a cyclic subgroup of G? (It is easiest to check if you come up with an algebraic description of L.) Be sure to justify your answers.

2. For each of the following, determine whether or not the indicated map ϕ is a homomorphism, and in the cases where ϕ is a homomorphism, determine $\ker \phi$.
   (a) $\phi\colon G \to G$, where G is any group, and $\phi(x) = x^{-1}$.
   (b) $\phi\colon \mathbb{Z} \to \mathbb{Z}$, where $\phi(n) = n - 1$.
   (c) $\phi\colon \mathbb{R}^* \to \mathbb{R}^*$, where $\phi(x) = |x|$.
   (d) $\phi\colon D_4 \to \mathbb{Z}_2$, where $\phi(\rho^i) = 0, \phi(\rho^i\tau) = 1$, for all $i, 0 \le i \le 3$.
   (e) $\phi\colon \mathbb{Z}_7 \to \mathbb{Z}_2$, where $\phi(\bar{x}) =$ the remainder of $x \mod 2$.

3. Construct an example of a non-trivial homomorphism between the two indicated groups, if this is possible, or explain why this is not possible. (Note: You do not have to prove that your map is a homomorphism.)
   (a) $\phi\colon \mathbb{Z}_3 \to \mathbb{Z}_5$
   (b) $\phi\colon \mathbb{Z}_4 \to \mathbb{Z}_{12}$
   (c) $\phi\colon \mathbb{Z}_{10} \to \mathbb{Z}_8$

4. Let G be any group, $a \in G$, and let $\phi\colon \mathbb{Z} \to G$ be the homomorphism defined by $\phi(n) = a^n$. Describe all the possibilities for $\ker \phi$.

5. Show that D_4 contains a subgroup which is isomorphic to the Klein 4-group.
6. Show that, in $\mathbb{C}^*$, the subgroup generated by i is isomorphic to $\mathbb{Z}_4$.
7. We define the *quaternion group* Q_8 as the subgroup of $SL_2(\mathbb{C})$ consisting of the subset $\{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, where each element is defined as follows:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

   Note that then, for example, $-\mathbf{i} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, and $|Q_8| = 8$. Prove that $D_4 \not\cong Q_8$.
8. Let $\phi : G \to H$ be a group homomorphism. If G is abelian, prove that $\phi(G)$ is abelian. Recall that $\phi(G) = \{h \in H \mid \exists\, g \in G \text{ such that } \phi(g) = h\}$. Does this prove that H is also abelian? Why or why not?
9. Let $\phi\colon G \to H$ be a group homomorphism. Prove that $\phi(G)$ is abelian if and only if for all $x, y \in G$, we have $xyx^{-1}y^{-1} \in \ker(\phi)$.