Exercises 15: 6, 11, 34,
Exercises 19: 9, 10, 23, 26, 29, 30

Additional exercises:

1. Recall the *norm* of a complex number $a+bi$ is defined to be $\sqrt{a^2+b^2}$. It is the length of the vector in the complex plane that represents $a+bi$. The purpose of this problem is to generalize this idea to the *quadratic field* $\mathbb{Q}[\sqrt{D}] = \{a+b\sqrt{D} \mid a,b \in \mathbb{Q}\}$, where $D \in \mathbb{Z}$ is fixed. For $\alpha = a+b\sqrt{D}$, define the *norm* of α to be $N(\alpha) = a^2 - b^2 D$.

   (a) Show that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in \mathbb{Q}(\sqrt{D})$.

   (b) Show that $N(\alpha) \in \mathbb{Z}$ for every $\alpha \in \mathbb{Z}[\sqrt{D}] = \{a+b\sqrt{D} \mid a,b \in \mathbb{Z}\}$.

   (c) Assume $D \equiv 1 \pmod 4$. Show that $N(\alpha) \in \mathbb{Z}$ for every

   $$\alpha \in \left\{ a + b\left(\frac{1+\sqrt{D}}{2}\right) \middle| a,b \in \mathbb{Z} \right\}.$$

   (d) Prove that:

      i. when $D \equiv 2 \pmod 4$ or $D \equiv 3 \pmod 4$, α is a unit in $\mathbb{Z}[\sqrt{D}]$ if and only if $N(\alpha) = \pm 1$

      ii. when $D \equiv 1 \pmod 4$, α is a unit in $\{a + b\left(\frac{1+\sqrt{D}}{2}\right) \mid a,b \in \mathbb{Z}\}$ if and only if $N(\alpha) = \pm 1$.

2. Recall that $\mathbb{Z}[i] = \{a+bi \mid a,b \in \mathbb{Z}\}$.

   (a) Show that $\mathbb{Z}[i]$ is an integral domain.

   (b) What is the smallest field containing $\mathbb{Z}[i]$? That is, find a field F such that $\mathbb{Z}[i] \subseteq F$ and such that for any field $F' \supseteq \mathbb{Z}[i]$, $F \subseteq F'$. For this problem, you only need to give an answer, not a proof.

3. Let X be an infinite set, and let $\mathcal{P}(X) = \{A \subseteq X\}$ be the power set of X (the set of all subsets of X). Define two operations on $\mathcal{P}(X)$ as follows:

   - $A+B = A \triangle B$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the *symmetric difference* of the sets A and B. (Recall that $A \setminus B = \{a \in A \mid a \notin B\}$.)
   - $A \cdot B = A \cap B$.

Prove that $(\mathcal{P}(X), +, \cdot)$ is a commutative ring with unity. Is $\mathcal{P}(X)$ a field? An integral domain? Find $\mathrm{char}(\mathcal{P}(X))$. (This is an example of an infinite ring with non-zero characteristic.)

4. Prove that every field is an integral domain.

5. Let R, S be non-trivial rings. Prove that the ring $R \times S$ is not an integral domain.