Bijection between $\mathbb{R}^3$ and $\mathbb{R}^2$

How can we find a bijection $g\colon \mathbb{R}^3 \to \mathbb{R}^2$? First, note that it is enough to find a bijection $f\colon \mathbb{R}^2 \to \mathbb{R}$, since then $g(x, y, z) = f(f(x, y), z)$ is automatically a bijection from $\mathbb{R}^3 \to \mathbb{R}$.

Next, note that since there is a bijection from $[0, 1] \to \mathbb{R}$, it is enough to find a bijection from the unit square $[0, 1]^2$ to the unit interval $[0, 1]$. It does not really matter whether we consider $[0, 1]$, $(0, 1]$, or $(0, 1)$, since there are easy bijections between all of these.

Mapping the unit square to the unit interval

There are a number of ways to proceed in finding a bijection from the unit square to the unit interval. One approach is to fix up an "interleaving" technique: writing $\langle 0.a_1a_2a_3..., 0.b_1b_2b_3...\rangle$ to $0.a_1b_2a_2b_2a_3b_3....$ This doesn't quite work, because there is a question of whether to represent $\frac{1}{2}$ as 0.5000... or as 0.4999.... We can't use both, since then $\langle 12, 0\rangle$ goes to both $\frac{1}{2} = 0.5000...$ and to $\frac{9}{22} = 0.40909...$ and we don't even have a function, much less a bijection. But if we arbitrarily choose to the second representation, then there is no element of $[0, 1]^2$ that is mapped to $\frac{1}{2}$, and if we choose the first there is no element that is mapped to $\frac{9}{22}$, so either way we fail to have a bijection.

This problem can be fixed.

First, we will deal with $(0, 1]$ rather than with $[0, 1]$; bijections between these two sets are well-known. For real numbers with two decimal expansions, such as $\frac{1}{2}$, we will agree to choose the one that ends with nines rather than with zeroes. So for example we represent $\frac{1}{2}$ as 0.4999....

Now instead of interleaving single digits, we will break each input number into chunks, where each chunk consists of some number of zeroes (possibly none) followed by a single non-zero digit. For example, $\frac{1}{200} = 0.00499...$ is broken up as 004 9 9 9... and 0.01003430901111... is broken up as 01 003 4 3 09 01 1 1 ...

This is well-defined since we are ignoring representations that contain infinite sequences of zeroes.

Now instead of interleaving digits, we interleave chunks. To interleave0.004999... and 0.01003430901111..., we get 0.004 01 9 003 9 4 9.... This is obviously reversible. It can never produce a result that ends with an infinite sequence of zeroes, and similarly the reverse mapping can never produce a number with an infinite sequence of trailing zeroes, so we win. A problem example similar to the one from a few paragraphs ago is resolved as follows: $\frac{1}{2} = 0.4999...$ is the unique image of $\langle 0.4999..., 0.999...\rangle$ and $\frac{9}{22} = 0.40909...$ is the unique image of $\langle 0.40909..., 0.0909...\rangle$.

(Taken from `https://math.stackexchange.com/questions/183361/examples-of-bijective-map-from-mathbbr3-rightarrow-mathbbr`)