1. Let G be a group and let
$$\mathrm{Inn}(G) = \{i_g\colon G \to G \mid g \in G\}$$
where $i_g\colon G \to G$ is the isomorphism defined by $i_g(x) = gxg^{-1}$ for all $x \in G$. It is true that $\mathrm{Inn}(G)$ is a group (try to prove this). Let
$$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G,$$
called the center of G. It is true that $Z(G) \triangleleft G$ (try to prove this).
Prove that
$$G/Z(G) \simeq \mathrm{Inn}(G).$$

2. Let $G = G_1 \times G_2$, and suppose $|G_1| = 10$ and $|G_2| = 15$. Prove that G is not cyclic.

3. Let G be a group of order 27, and suppose there is an element $a \in G$ such that $a^9 \neq e$. Prove that G is cyclic.

4. Let G be a group and H a subgroup of G such that $|G : H| = 2$. Prove that H is a normal subgroup of G.

5. Describe all homomorphisms between the following groups:
   (a) $S_3 \to \mathbb{Z}_3$
   (b) $\mathbb{Z}_4 \to \mathbb{Z}_6$

6. Determine the isomorphism type of the group $\mathbb{Z}_4 \times \mathbb{Z}_8/\langle(\bar{1}, \bar{2})\rangle$. Hint: consider the element $(\bar{0}, \bar{1}) + H$, where $H = \langle(\bar{1}, \bar{2})\rangle$.

7. Let G be a group and let
$$\mathrm{Aut}(G) = \{\sigma\colon G \to G \mid \sigma \text{ is an isomorphism}\}.$$
This is a group (try to prove this), called the automorphism group of G. A subgroup $H \leq G$ is called *characteristic in* G if $\sigma(H) \subseteq H$ for all $\sigma \in \mathrm{Aut}(G)$.
   (a) Prove that if H is characteristic in G, then $\sigma(H) = H$ for all $\sigma \in \mathrm{Aut}(G)$.
   (b) Let $Z(G)$ be the center of G, as in problem 1. Prove that $Z(G)$ is characteristic in G.

8. Let G_1, G_2, and H be groups, and let $f_1\colon G_1 \to H$ and $f_2\colon G_2 \to H$ be homomorphisms. Define

$$M = \{(a_1, a_2) \in G_1 \times G_2 \mid f_1(a_1) = f_2(a_2)\} \subset G_1 \times G_2.$$

Prove that M is a subgroup of $G_1 \times G_2$.

9. Prove that a group G is cyclic if and only if there is a surjective homomorphism $\phi\colon Z \to G$.