1. Write the element $\sigma = (1\,9\,3)(2\,5\,8\,4\,7\,6) \in S_9$ as the product of transpositions. Is $\sigma \in A_9$?

2. Find the order of $(1\,3\,5\,4\,2)(1\,2\,4\,5\,3)(1\,4\,)(2\,5)$ in S_5.

3. Find all possible non-trivial homomorphisms between the given groups.

   (a) $\phi\colon S_3 \to \mathbb{Z}_6$.

   (b) $\psi\colon \mathbb{Z}_8 \to \mathbb{Z}_6$.

4. Let ϕ be a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_6$ such that $\ker\phi = \{\bar{0}, \bar{6}, \bar{12}, \bar{18}, \bar{24}\}$. Prove that ϕ is surjective, and find all possibilities for $\phi(\bar{1})$.

5. Let $\phi\colon G \to G'$ be a homomorphism where $|G| = 9$. Find $|\ker(\phi)|$ if ϕ is:

   (a) trivial

   (b) injective

   (c) neither

6. Consider the dihedral group D_6. Show that:

   (a) $\langle \rho^3 \rangle \lhd D_6$, and

   (b) $D_6/\langle \rho^3 \rangle \simeq S_3$.

7. Is $\langle (1\,2\,3\,) \rangle$ a normal subgroup of S_4?

8. An *automorphism* of a group G is an isomorphism from G to itself. Let G be an abelian group. Show that the map $\phi\colon G \to G$ defined by $\phi(x) = x^{-1}$ for all $x \in G$ is an automorphism of G.

9. Let G be a group, $H \lhd G$, and ϕ an automorphism of G. Show that $\phi(H)$ is a normal subgroup of G.

10. Let $Z(G)$ be the *center* of a group G. Recall that $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$ (your book may have used a different notation for this group). You showed in homework that $Z(G) \leq G$ (it would be a good idea to review this proof before starting this problem). Show that:

    (a) $Z(G) \lhd G$, and

    (b) if $G/Z(G)$ is cyclic, then G is abelian.

11. Let $H, K \leq G$. Then $HK = \{hk \mid h \in H, k \in K\}$. Show that HK is a subgroup of G if and only if $HK = KH$.

12. Let $K \lhd G$ and let H be a subgroup of G. Show that $K \cap H \lhd H$.

13. Let G be a group with $|G| = p^2$, where p is prime. Show that every proper subgroup of G is cyclic.

14. Show that the non-zero elements of $\mathbb{Z}_p$, where p is prime, form a group under multiplication.

15. For each group G, find two non-trivial proper subgroups $H, K \leq G$ such that $G \simeq H \times K$, or explain why it is not possible.

    (a) $\mathbb{Z}_{10}$

    (b) $\mathbb{Z}_{15}$

    (c) $\mathbb{Z}_{16}$

    (d) $\mathbb{Z}_{36}$

16. Let $G = \mathbb{Z}_6 \times \mathbb{Z}_8$ and define a map $\phi\colon G \to \mathbb{Z}_3 \times \mathbb{Z}_4$ by $\phi((h_1, h_2)) = (h_1 \pmod 3, h_2 \pmod 4))$ for any $h_1 \in \mathbb{Z}_6$ and $h_2 \in \mathbb{Z}_8$.

    (a) Show that ϕ is a homomorphism.

    (b) Find $\ker(\phi)$.

    (c) Find $\phi(G)$.

17. Let $G = \mathbb{Z}_{10} \times \mathbb{Z}_{12}$ and let $H = \langle(\bar{5}, \bar{2})\rangle$.

    (a) Find $|G/H|$.

    (b) Use the fundamental theorem of finitely generated abelian groups to list all the possible isomorphism types for G/H.

    (c) Find the order of the element $(\bar{3}, \bar{3}) + H$ in G/H.

    (d) Which group in (b) is isomorphic to G/H?