Angenent

math 234 — the first midterm — lecture 2

october 9, 2013

your name: ______________________

your ta & discussion: (circle one)

Peter Mueller   Lakshmi Priya   Chad Weibel

Hans Chaumont   Guru Subramani   Rohit Nagpal

problem	**score**
1	
2	
3	
4	
5	
6	
total	

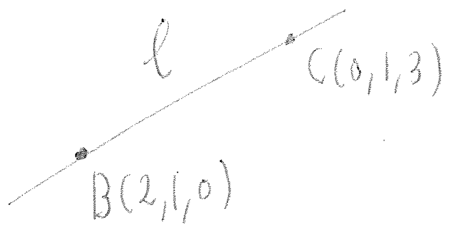
1. (a) [7 points] Let $\mathcal{P}$ the plane with equation $z = 2x + 3y - 5$. What is the distance from the point $A(1,1,1)$ to the plane $\mathcal{P}$?

$P(1,1,0)$ is on the plane $\mathcal{P}$. $\vec{n} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ is a normal vector of $\mathcal{P}$.

(Note that the plane equation is $2x + 3y - z = 5$)

$$\text{dist}(A, \mathcal{P}) = \frac{|\vec{AP} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{\left| \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right|}{\sqrt{4+9+1}} = \frac{1}{\sqrt{14}}$$

(b) [8 points] Represent the line ℓ through the points $B(2,1,0)$ and $C(0,1,3)$ as a parametric curve. Let P be a point on the line ℓ and consider the line segment AP, where A is again the point $A(1,1,1)$. Where should we choose P on ℓ to make sure that AP is perpendicular to ℓ?



$\vec{BC} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ is parallel to ℓ

B is on ℓ. A parametrization for ℓ is $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ $\quad -\infty < t < \infty$

$P(2-2t, 1, 3t)$ denotes an arbitrary point on ℓ.

We want $\vec{AP} \perp \ell$, which is same as requiring $\vec{AP} \perp \vec{BC}$

$$0 = \vec{AP} \cdot \vec{BC} = \begin{pmatrix} 1-2t \\ 0 \\ 3t-1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = 4t - 2 + 9t - 3 = 13t - 5.$$

Thus, choose $t = 5/13$ and $P\left(\frac{16}{13}, 1, \frac{15}{13}\right)$.

2. A point $X(t)$ moves around in the plane. At time t its position vector is $\vec{x}(t) = \begin{pmatrix} 3t - t^2 \\ t^2 - 1 \end{pmatrix}$.

(a) [7 points] Compute the velocity and acceleration of the point.

velocity $= \vec{v}(t) = \vec{x}'(t) = \begin{pmatrix} 3 - 2t \\ 2t \end{pmatrix}$

acceleration $= \vec{a}(t) = \vec{v}'(t) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

(b) [7 points] Where does the tangent line to the curve at the point $X(t)$ with $t = 1$ intersect the y axis?

The tangent line to $X(t)$ at $t=1$ can be parametrized as

$\vec{x}(1) + s\,\vec{v}(1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This line intersects the y-axis

when the x-coordinate ~~$0 + 2s = 0$~~ $2 + s = 0$, $s = -2$

The intersection point with the y-axis is $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.

[Problem 1, continued]

(c) [7 points] At which time are velocity and acceleration perpendicular?

$$0 = \vec{v}(t) \cdot \vec{a}(t) = \begin{pmatrix} 3-2t \\ 2t \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -6 + 4t + 4t = 8t - 6$$

At $t = \frac{3}{4}$, the velocity and the acceleration vectors are perpendicul

(d) [7 points] Write an integral for the length of the path traced out by $X(t)$

The length of the path traced out between $t = a$ and $t = b$ (where $a < b$) is $\int_a^b \|v(t)\| \, dt = \int_a^b \sqrt{(3-2t)^2 + 4t^2} \, dt$

3. [8 points] We consider an object moving through space. Let its position vector at time t be $\vec{x}(t)$. Suppose that the velocity vector of the object is always perpendicular to the position vector. Show that $\|\vec{x}(t)\|$ does not depend on t.

It suffices to check that $\|\vec{x}(t)\|^2 = \vec{x}(t) \cdot \vec{x}(t)$ does not depend on t. For this, we compute

$$\frac{d}{dt}\left(\vec{x}(t) \cdot \vec{x}(t)\right) \overset{\text{(product rule)}}{=} 2\vec{x}(t)\ \vec{x}'(t)$$

$$= 2\vec{x}(t) \cdot \vec{v}(t)$$

$$= 0$$ because $\vec{x}(t) \perp \vec{v}(t)$ by assumption.

We conclude that

$\|\vec{x}(t)\|^2$ is constant.

4. For each of the following functions decide if they are positive definite, negative definite, indefinite, or semidefinite quadratic forms. Also *draw the zero set* of the function, and indicate *the region where the function is positive,* and *the region where the function is negative.*

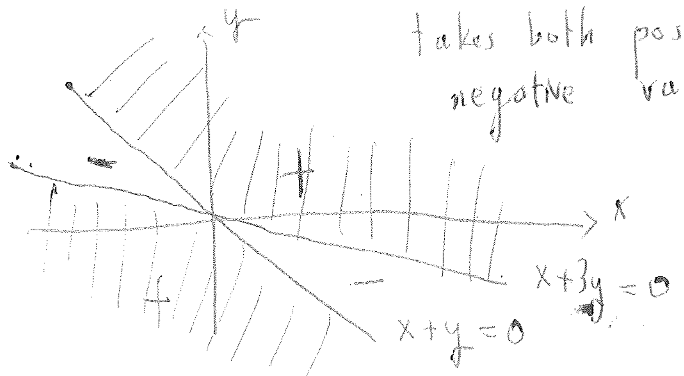
(a) [7 points] $f(x, y) = x^2 + 4xy + 3y^2$

$4AC - B^2 = 4 \cdot 1 \cdot 3 - 4^2 = -4 < 0$

$A=1,\ B=4,\ C=3$

$f(x,y) = (x+3y)(x+y)$

f is indefinite, f takes both positive and negative values.

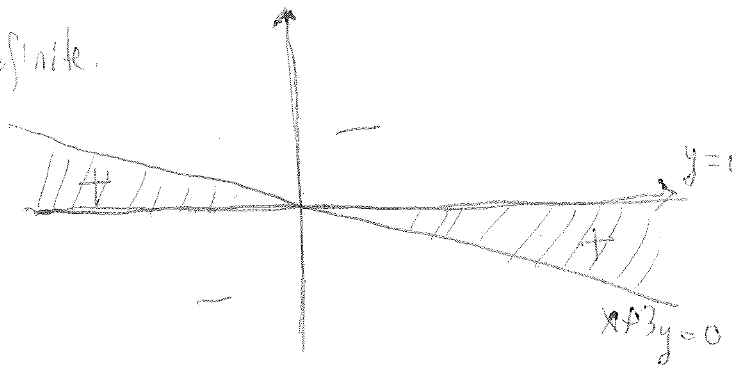


(b) [7 points] $g(x, y) = -xy - 3y^2$

$A=0,\ B=-1,\ C=-3$

$4AC - B^2 = -1 < 0$ g is indefinite.

$g(x,y) = -y\,(x+3y)$



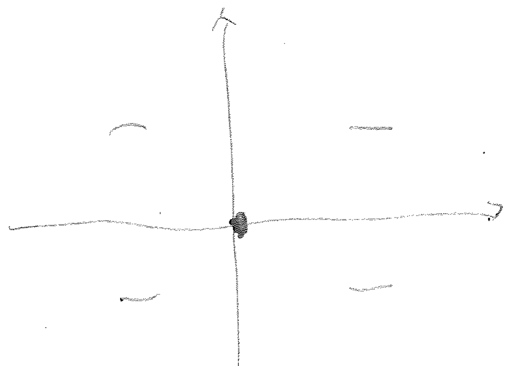
(c) [7 points] $h(x, y) = -2x^2 + 4xy - 6y^2$

$A = -2$
$B = 4$
$C = -6$

$4AC - B^2 = 48 - 16 = 32 > 0$

$A < 0$ h is negative-definite.

$h < 0$ everywhere except at $(0,0)$ where it takes the value 0.



5. (a) [7 points] $f(x,y) = \dfrac{xy}{2y+\cos(x)}$; compute f_x and f_y.

$$f_x = \frac{y}{2y+\cos x} + \frac{xy \cdot \sin x}{(2y+\cos x)^2}$$

$$f_y = \frac{x}{2y+\cos(x)} - \frac{2xy}{(2y+\cos x)^2}$$

where we used the product rule.

(b) [8 points] Find the equation of the tangent plane to the graph of $z = f(x,y) = 2x^3 - \frac{1}{4}xy^2$ at the point with $x = 1, y = 2$. Then explain how you can use the linear approximation formula to approximate $f(0.95, 2.1)$.

Let us answer the second question only. By the linear approximation formula,

$$f(0.95, 2.1) \approx f(1,2) + (-0{,}05)\, f_x(1,2) + 0{,}1\, f_y(1,2)$$

$$f(1,2) = 2 \cdot 1^3 - \frac{1}{4} \cdot 1 \cdot 2^2 = 1$$

$$f_x(x,y) = 6x^2 - y^2/4 \qquad f_x(1,2) = 5$$

$$f_y(x,y) = -\frac{xy}{2} \qquad f_y(1,2) = -1$$

$$\approx 1 - 0{,}25 - 0{,}1 = 0{,}65$$

6. (a) [7 points] Let g be a function of one variable for which we know these values:

t	0	$\frac{1}{2}$	1
$g(t)$	1	0	-1
$g'(t)$	0	2	1

If $f(x,y) = g(\frac{x}{x+y})$ then compute $\frac{\partial f}{\partial x}(1,1)$ and $\frac{\partial f}{\partial y}(0,4)$.

Chain rule

$f(x,1) = g\left(\frac{x}{x+1}\right)$, so $\frac{\partial f}{\partial x}(x,1) = g'\left(\frac{x}{x+1}\right)\left[\frac{1}{x+1} - \frac{x}{(x+1)^2}\right]$

Set $x = 1$. $\frac{\partial f}{\partial x}(1,1) = g'\left(\frac{1}{2}\right) \frac{1}{4} = 2 \cdot \frac{1}{4} = \frac{1}{2}$.

(use table)

Similarly,

$f(0,y) = g\left(\frac{0}{y}\right) = g(0) = 1$ (const)

$\frac{\partial f}{\partial y}(0,y) = 0$ Set $y = 4$. $\frac{\partial f}{\partial y}(0,4) = 0$.

(b) [7 points] If $z = h(x,y)$ is a function of two variables for which we know

$h(2,0) = 1, h_x(2,0) = A, h_y(2,0) = B,$ (A and B are constants)

then compute $f'(0)$ where $f(t)$ is the function of one variable defined by

$f(t) = h\big(2 + 3t, \sin(\pi t)\big)$.