Angenent Solutions

① $f(x,y) = xe^{x-2y}$

(a) $\vec{\nabla} f(x,y) = \langle e^{x-2y} + xe^{x-2y},\ -2xe^{x-2y} \rangle$, $f(2,1) = 2e^0 = 2$

$\vec{\nabla} f(2,1) = \langle e^0 + 2e^0, -4e^0 \rangle = \langle 3,-4 \rangle$

$L(x,y) = f(2,1) + \vec{\nabla} f(2,1) \cdot \langle x-2, y-1 \rangle$

$$\boxed{L(x,y) = 2 + 3(x-2) - 4(y-1)}$$

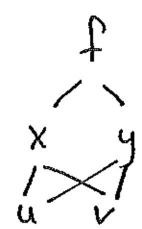
(b) $f(1.99, 1.02) \approx 2 + 3(-0.01) - 4(0.02)$

$\approx 2 - 0.03 - 0.08$

$\approx 2 - 0.11$

$\approx \boxed{1.89}$

② $g(u,v) = f(u \ln v, u+v)$: $x = u \ln v$, $y = u+v$



(a) $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial u}$

$$\boxed{\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}(\ln v) + \frac{\partial f}{\partial y}}$$

(b) $\dfrac{\partial^2 g}{\partial u \partial v} = \dfrac{\partial^2 g}{\partial v \partial u} = \dfrac{\partial}{\partial v}\left(\dfrac{\partial g}{\partial u}\right) = \dfrac{\partial}{\partial v}\Big(\overbrace{\dfrac{\partial f}{\partial x} \cdot \ln v}^{\text{product rule}}\Big) + \dfrac{\partial}{\partial v}\left(\dfrac{\partial f}{\partial y}\right)$

$= \left[\left(\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial x}{\partial v}\right)(\ln v) + \dfrac{\partial f}{\partial x} \cdot \dfrac{1}{v} + \left(\dfrac{\partial^2 f}{\partial y \partial x} \cdot \dfrac{\partial y}{\partial v}\right)(\ln v)\right] + \left[\dfrac{\partial^2 f}{\partial x \partial y} \cdot \dfrac{\partial x}{\partial v} + \dfrac{\partial^2 f}{\partial y^2} \cdot \dfrac{\partial y}{\partial v}\right]$

$$\boxed{\frac{\partial^2 g}{\partial u \partial v} = \frac{\partial^2 f}{\partial x^2} \cdot \frac{u}{v} \cdot \ln v + \frac{\partial f}{\partial x} \cdot \frac{1}{v} + \frac{\partial^2 f}{\partial y \partial x} \cdot \ln v + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{u}{v} + \frac{\partial^2 f}{\partial y^2}}$$

③ (a) $f(x,y) = x^2 - 3y^2 + y^3$

$f_x(x,y) = 2x = 0 \Rightarrow x = 0$ $\boxed{(0,0),\ (0,2)}$

$f_y(x,y) = -6y + 3y^2 = 0$

$3y(y-2) = 0$

$y = 0$ or $y = 2$

(b)

	$(0,0)$	$(0,2)$
$f_{xx}(x,y) = 2$	2	2
$f_{xy}(x,y) = 0$	0	0
$f_{yy}(x,y) = -6 + 6y$	-6	$-6+12 = 6$
$f_{xx}f_{yy} - f_{xy}^2$	$-12 < 0$ SADDLE	$12 > 0$ and $f_{xx} > 0$ MINIMUM

Tangent line to level set: Since the level set intersects itself at a saddle pt, the tangent line is not defined.

④ $f(x,y) = xy^2$ on $x^2 + y^2 = 3 = g(x,y)$

$\vec{\nabla} f(x,y) = \langle y^2, 2xy \rangle$ $\vec{\nabla} g(x,y) = \langle 2x, 2y \rangle$

$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ x^2 + y^2 = 3 \end{cases}$ or $\left.\begin{cases} \vec{\nabla} g = 0 \longrightarrow x = 0, y = 0 \\ x^2 + y^2 = 3 \longrightarrow 0^2 + 0^2 \neq 3 \end{cases}\right\}$ no solution

$y^2 = \lambda 2x \longrightarrow x = 0$ or $\lambda = \frac{y^2}{2x}$ $\underline{x=0}$: $y = \pm\sqrt{3} \rightarrow$ doesn't satisfy $\vec{\nabla} f = \vec{\nabla} g$

$2xy = \lambda 2y$

$\underline{\lambda = \frac{y^2}{2x}}$: $2xy = \frac{y^2}{\cancel{2x}} \cdot \cancel{2}y$

$2x^2y - y^3 = 0$

$y(2x^2 - y^2) = 0$

$y = 0$ or $y^2 = 2x^2$

$x = \pm\sqrt{3}$ $x^2 + 2x^2 = 3$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

$y^2 = 2$

$y = \pm\sqrt{2}$

$(\pm 1, \pm\sqrt{2})$

~~$(\pm\sqrt{3}, 0)$~~ $\rightarrow$ doesn't satisfy $\vec{\nabla} f = \vec{\nabla} g$

(x,y)	$f(x,y) = xy^2$
$(1,\sqrt{2})$	2
$(1,-\sqrt{2})$	$+2$
$(-1,\sqrt{2})$	-2
$(-1,-\sqrt{2})$	-2

$$\boxed{\begin{array}{l}\text{Maximum at } (1,\sqrt{2}),\ (1,-\sqrt{2}),\ \&\ (-1,\sqrt{2}) \\ \text{minimum at } (-1,-\sqrt{2})\end{array}}$$

⑤ $z = \dfrac{1}{(x+y)^2}$, $3 \le x \le 6$, $0 \le y \le 2$

(a) $\displaystyle\int_3^6 \int_0^2 \int_0^{\frac{1}{(x+y)^2}} dz\,dy\,dx$

OR

$\displaystyle\int_3^6 \int_0^2 \frac{1}{(x+y)^2}\,dy\,dx$

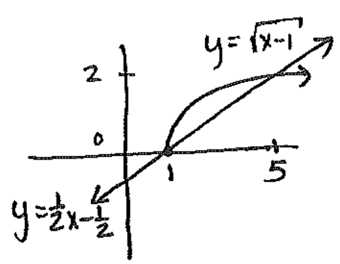
(b) $\displaystyle\int_3^6 \int_0^2 \frac{1}{(x+y)^2}\,dy\,dx = \int_3^6 \frac{-1}{(x+y)}\Big|_0^2 dx = \int_3^6 \frac{-1}{(x+2)} + \frac{1}{x}\,dx$

$= \left(-\ln|x+2| + \ln|x|\right)\Big|_3^6 = -\ln 8 + \ln 6 + \ln 5 - \ln 3$

$= \ln(30) - \ln(24) = \ln\left(\frac{30}{24}\right) = \boxed{\ln(5/4)}$

⑥ $I = \displaystyle\iint_R f(x,y)\,dA = \int_1^5 \int_{\frac{x-1}{2}}^{\sqrt{x-1}} f(x,y)\,dy\,dx$

(a)



$y = \frac{1}{2}x - \frac{1}{2}$

$y = \sqrt{x-1}$

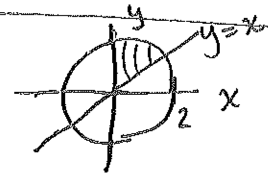
(b) $I = \displaystyle\int_0^2 \int_{y^2+1}^{2y+1} f(x,y)\,dx\,dy$

$y = \sqrt{x-1} \rightsquigarrow x = y^2+1$

$y = \frac{x-1}{2} \rightsquigarrow x = 2y+1$

⑦ $x\geq 0,\ y\geq x,\ x^2+y^2\leq 4,\ 0\leq z\leq x^2+y^2$

(a) $\boxed{0\leq z\leq r^2,\quad 0\leq r\leq 2,\ \frac{\pi}{4}\leq\theta\leq\frac{\pi}{2}}$



(b) Average of z in R

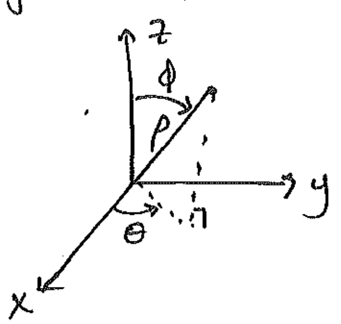
$$\text{Average of } z = \frac{\iiint_R z\,dV}{\iiint_R 1\,dV}$$

$$\iiint_R 1\,dV = \int_{\frac{\pi}{4}}^{\pi/2}\int_0^2\int_0^{r^2} r\,dz\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_0^2 r^3\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left.\frac{r^4}{4}\right|_0^2 d\theta = \int_{\pi/4}^{\pi/2} 4\,d\theta = 4(\pi/2-\pi/4) = \boxed{\pi}$$

⑧ $x^2+y^2+z^2\leq 9,\quad x\geq 0, y\geq 0,\ z\geq 0,\ y\geq x$

(a)



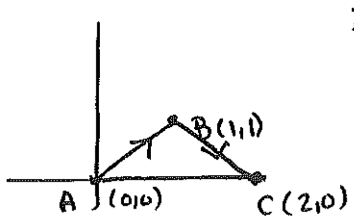
$\underbrace{0\leq\rho\leq 3}_{\text{the sphere.}},\ \underbrace{0\leq\phi\leq\pi/2}_{\text{above xy-plane i.e. } z\geq 0},\ \underbrace{\frac{\pi}{4}\leq\theta\leq\frac{\pi}{2}}_{\text{b/c in xy-plane:}}$

(b) $\boxed{I=\iiint_R x\,dV = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\pi/2}\int_0^3 \rho\sin\phi\cos\theta\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$

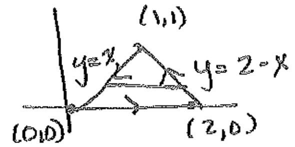
$x = r\cos\theta = \rho\sin\phi\cos\theta$

⑨ (a) $\vec{F}(x,y) = \langle -1, 1\rangle$. $\vec{F}$ is conservative: If $f(x,y) = y-x$, $\nabla f = \vec{F}$. So by FTC of line integrals,

$$\int_C \vec{F}\cdot\vec{T}\,ds = f(2,0) - f(0,0) = -2-0 = \boxed{-2}$$



(c) $\vec{G}(x,y) = \langle \underset{P}{0}, \underset{Q}{x} \rangle$



$Q_x = 1$, $P_y = 0$

By Green's Thm,

$$\oint_T \vec{G}\cdot\vec{T}\,ds = \iint_R (1-0)\,dA = \int_0^1 \int_y^{2-y} dx\,dy = \int_0^1 2-y-y\,dy$$

$$= \int_0^1 2-2y\,dy = 2y - y^2 \Big|_0^1 = 2-1 = \boxed{1}$$

<u>OR</u> Since $Q_x - P_y = 1-0 = 1$, the integral $\iint_R 1\,dA$ = area of T

$$= 2\cdot 1\cdot \tfrac{1}{2} = \boxed{1}$$