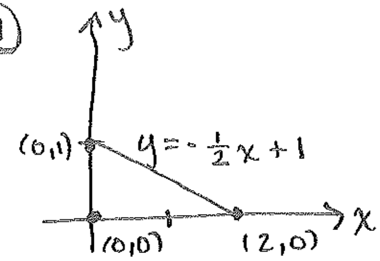
Final Review Part II Solutions

① 

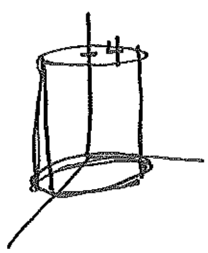
$$\int_0^2 \int_0^{-\frac{1}{2}x+1} (x+y)\,dy\,dx$$

$$= \int_0^2 xy + \frac{y^2}{2} \Big|_{y=0}^{y=-\frac{1}{2}x+1} dx$$

$$= \int_0^2 -\frac{1}{2}x^2 + x + \frac{1}{2}\left(\frac{1}{4}x^2 - x + 1\right) dx$$

$$= \int_0^2 \left(-\frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{2}\right) dx$$

$$= -\frac{1}{8}x^3 + \frac{x^2}{4} + \frac{1}{2}x \Big|_0^2 = -1+1+1-0 = \boxed{1}$$

② 

$$\int_0^{2\pi}\int_0^5\int_0^4 e^{r^2}\, r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^5 zre^{r^2}\Big|_0^4 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^5 4re^{r^2}\,dr\,d\theta \quad \begin{pmatrix} u = r^2 \\ du = 2r\,dr \end{pmatrix}$$

$$= \int_0^{2\pi}\int_0^{25} 2e^u\,du\,d\theta = \int_0^{2\pi} 2e^u\Big|_0^{25} d\theta = \int_0^{2\pi} 2e^{25} - 2\,d\theta$$

$$= \boxed{4\pi(e^{25}-1)}$$

③ $\int_C F\cdot ds = \int_a^b F(\gamma(t))\cdot\gamma'(t)\,dt$

$\gamma(t) = \begin{pmatrix} t \\ 2-t-t^2 \end{pmatrix}, \quad t\in[-2,1]$

$\gamma'(t) = \begin{pmatrix} 1 \\ -1-2t \end{pmatrix}$

$$\int_C F\cdot ds = \int_{-2}^1 \begin{pmatrix} 2te^{t^2-1}\cos(2-t-t^2) \\ -e^{t^2-1}\sin(2-t-t^2) \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -1-2t \end{pmatrix} dt$$

This is really ugly, so let's check if there's an easier way: Is F conservative?

If $F = \nabla f$, then

$f_x = 2xe^{x^2-1}\cos y$ & $f_y = -e^{x^2-1}\sin y$

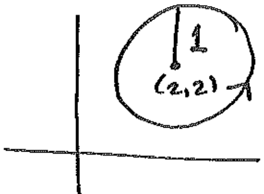
So, $f(x,y) = e^{x^2-1}\cos y$.

Thus, $\int_C F\cdot ds = f(1,0) - f(-2,0)$

$$= e^0\cos 0 - e^3\cos 0$$

$$= \boxed{1-e^3}$$

4 Directly:



$$\oint_C (x^6+3y)\,dx + (2x-e^y)\,dy$$

$$= \oint_C F\cdot ds, \text{ where } F = \begin{pmatrix} x^6+3y \\ 2x-e^y \end{pmatrix}$$

$$\gamma(t) = \begin{pmatrix} 2+\cos t \\ 2+\sin t \end{pmatrix}$$

$$\gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

$t\in[0,2\pi]$

$$= \int_0^{2\pi} \begin{pmatrix} (2+\cos t)^6 + 3(2+\sin t) \\ 2(2+\cos t) - e^{(2+\sin t)} \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

$$= \int_0^{2\pi} \Big[-\sin t\,(2+\cos t)^6 - 6\sin t - 3\sin^2 t + 4\cos t + 2\cos^2 t - \cos t\, e^{(2+\sin t)}\Big] dt$$

$$= \left(\frac{1}{7}(2+\cos t)^7 + 6\cos t\right)\Big|_0^{2\pi} - 3\int_0^{2\pi} \frac{1}{2} - \frac{\cos 2t}{2}\,dt + 4\sin t\Big|_0^{2\pi}$$

$$+ 2\int_0^{2\pi} \frac{1}{2} + \frac{\cos 2t}{2}\,dt + e^{2+\sin t}\Big|_0^{2\pi}$$

$$= \cancel{\frac{1}{7}\cdot 3^7 + 6 - \frac{1}{7}3^7 - 6} - 3\cdot\frac{1}{2}\cdot 2\pi + \frac{3\sin 2t}{4}\Big|_0^{2\pi} + 0 - 0 + 2\cdot\frac{1}{2}\cdot 2\pi$$

$$+ \frac{2\sin(2t)}{4}\Big|_0^{2\pi} + \cancel{e^2 - e^2}$$

$$= -3\pi + 2\pi$$

$$= \boxed{-\pi}$$

Green's Thm:

$$\oint_C F \cdot ds = \iint_A (Q_x - P_y)\, dA$$

$$= \iint_A (2-3)\, dA$$

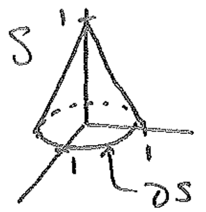
$$= \iint_A -1\, dA$$

$$= -1 \text{ (area of circle)}$$

$$= -1(\pi \cdot 1^2)$$

$$= \boxed{-\pi}$$

⑤ By Stoke's thm, $\iint_S \nabla\times F\cdot n\,d\sigma = \int_{\partial S} F\cdot ds$



∂S is a circle of radius 1 in the xy-plane:

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \quad t\in[0,2\pi] \quad \text{ccw (by right hand rule)}$$

$$\gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}$$

$$\int_0^{2\pi} \begin{pmatrix} \sin t \\ 0 \\ \cos(\cos^2 t) \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt$$

$$= \int_0^{2\pi} -\sin^2 t\,dt = \int_0^{2\pi} \left(-\frac{1}{2} + \frac{\cos 2t}{2}\right) dt$$

$$= -\frac{1}{2}\cdot 2\pi + \left.\frac{\sin(2t)}{4}\right|_0^{2\pi} = \boxed{-\pi}$$

⑥ the sphere is a closed surface, so we can use the divergence thm:

$\nabla\cdot F = y+1+1-2y+y+1 = 3$

So $\iint_S F\cdot n\,d\sigma = \iiint_R 3\,dV = 3\,(\text{volume of sphere})$

$$= 3\left(\frac{4}{3}\pi(2)^3\right) = \boxed{32\pi}$$

* without the divergence thm, this is really messy!

(7) $\gamma'(t) = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$

$$\int_0^1 \begin{pmatrix} e^{t^2} \\ te^{t^2} \\ (t^3+1)e^{t^3} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} dt = \int_0^1 \left(e^{t^2} + 2t^2 e^{t^2} + 3t^2(t^3+1)e^{t^3}\right) dt.$$

But this isn't an integral we know how to compute. So, check to see if F is conservative.

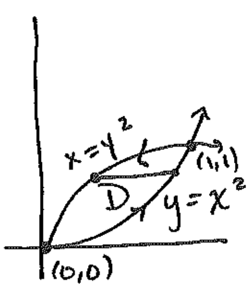
If $F = \nabla f$, then

$f_x = e^y$, $f_y = xe^y$, & $f_z = (\underbrace{z+1}_{u})\underbrace{e^z}_{dv}$ $du = 1\,dz$, $v = e^z$

$$f(x,y,z) = xe^y + g(y,z) = xe^y + h(x,z) = \underbrace{(z+1)e^z - e^z}_{= ze^z} + k(x,y)$$

So $f(x,y,z) = xe^y + ze^z$

$$\int_C F \cdot ds = f(\gamma(1)) - f(\gamma(0)) = f(1,1,1) - f(0,0,0) = \boxed{2e}$$

(8) 

$F = \begin{pmatrix} y + e^{\sqrt{x}} \\ 2x - \cos(y^2) \end{pmatrix}$. Green's thm:

$$\int_C F \cdot ds = \iint_D (Q_x - P_y)\, dx\,dy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2-1)\, dx\,dy = \int_0^1 (\sqrt{y} - y^2)\, dy$$

$$= \frac{2}{3}y^{3/2} - \frac{1}{3}y^3 \Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

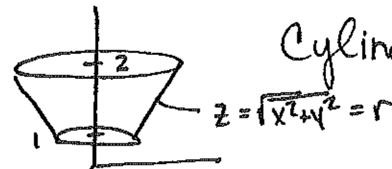
(12) (a) False (b) True (c) True (d) False (e) False

(13) $F = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \nabla f$, so F is conservative.

$$\int_C F \cdot ds = f(\gamma(\pi/2)) - f(\gamma(0))$$
$$= f(\pi/2, \pi/2, \pi/2) - f(0,0,0)$$
$$= e^0\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}\right) - e^0\left(0 + \tfrac{\pi}{2}\right)$$
$$= \pi - \pi/2$$
$$= \boxed{\pi/2}$$

(14) By the divergence thm, since the surface is closed,

$$\int_S F \cdot n\, d\sigma = \iiint_R (y^2 + x^2)\, dV$$



Cylindrical

$$= \int_0^{2\pi}\int_0^1\int_1^2 r^3\, dz\, dr\, d\theta + \int_0^{2\pi}\int_1^2\int_r^2 r^3\, dz\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^1 zr^3\Big|_1^2 dr\, d\theta + \int_0^{2\pi}\int_1^2 zr^3\Big|_r^2 dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^1 (2r^3 - r^3)\, dr\, d\theta + \int_0^{2\pi}\int_1^2 (2r^3 - r^4)\, dr\, d\theta$$
$$= \int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 d\theta + \int_0^{2\pi} \frac{r^4}{2} - \frac{r^5}{5}\Big|_1^2 d\theta = \int_0^{2\pi} \frac{1}{4}\, d\theta + \int_0^{2\pi}\left(8 - \frac{32}{5} - \frac{1}{2} + \frac{1}{5}\right) d\theta$$
$$= \frac{\pi}{2} + \frac{13}{10}\cdot 2\pi = \boxed{\frac{31\pi}{10}}$$

(15) $\gamma(t) = \begin{pmatrix} 3\cos t \\ 3\sin t \\ 1 - 3\cos t - 3\sin t \end{pmatrix}$, $t \in [0, 2\pi]$

Stokes' Thm: $C(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 1 - r\cos\theta - r\sin\theta \end{pmatrix}$, $C_r \times C_\theta = \begin{pmatrix} r \\ r \\ r^2 \end{pmatrix}$

$r \in [0,3]$, $\theta \in [0, 2\pi]$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ -yx^2 & y^2 z & z^2 \end{vmatrix} = \begin{pmatrix} -y^2 \\ 0 \\ x^2 \end{pmatrix}$$

$$\int_C F \cdot ds = \int_0^{2\pi}\int_0^3 \begin{pmatrix} -r^2\sin^2\theta \\ 0 \\ +r^2\cos^2\theta \end{pmatrix} \cdot \begin{pmatrix} r \\ r \\ r^2 \end{pmatrix} dr\, d\theta = \int_0^{2\pi}\int_0^3 -r^3(\sin^2\theta - \cos^2\theta)\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^3 r^3 \cos 2\theta\, dr\, d\theta$$
$$= \int_0^{2\pi} \frac{3^4}{4}\cos 2\theta\, d\theta = \frac{3^4}{8}\sin 2\theta\Big|_0^{2\pi} = \boxed{0}$$

⑨ By Stokes' Thm, $\int_S \nabla \times F \cdot n\,d\sigma = \int_C F \cdot ds$

$C: \gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \quad \gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}, \quad 0 \le t \le 2\pi$

$$\int_0^{2\pi} \begin{pmatrix} \sqrt{\cos t} \\ 3\cos t - e^{\sin t} \\ \cos^2 t - \sin^3 t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt$$

$$= \int_0^{2\pi} \Big(\underbrace{-\sin t (\cos t)^{1/2}}_{u = \cos t} + \underbrace{3\cos^2 t}_{= 3(\frac{1}{2} + \frac{1}{2}\cos(2t))} - \underbrace{\cos t\, e^{\sin t}}_{u = \sin t} \Big) dt$$

$$= \frac{2}{3}(\cos t)^{3/2} + \frac{3}{2}t + \frac{3}{4}\sin(2t) - e^{\sin t} \Big|_0^{2\pi}$$

$$= \left(\frac{2}{3}\cdot 1 - \frac{2}{3}\cdot 1\right) + (3\pi - 0) + (0 - 0) - (1 - 1)$$

$$= \boxed{3\pi}$$

⑩ Method 1: Directly

$C(s,t) = \begin{pmatrix} s\cos t \\ s\sin t \\ s^2 \end{pmatrix} \quad \begin{matrix} s \in [0,1] \\ t \in [0, 2\pi] \end{matrix} \qquad C_s = \begin{pmatrix} \cos t \\ \sin t \\ 2s \end{pmatrix} \quad C_t = \begin{pmatrix} -s\sin t \\ s\cos t \\ 0 \end{pmatrix}$

$$C_s \times C_t = \begin{vmatrix} i & j & k \\ \cos t & \sin t & 2s \\ -s\sin t & s\cos t & 0 \end{vmatrix} = i(-2s^2\cos t) - j(2s^2 \sin t) + k(s)$$

$$\int_0^{2\pi}\int_0^1 \begin{pmatrix} e^{s\sin t} + s\cos t \\ e^{\sin s^2} + \sin(s\cos t) \\ -s^2 + s^2 \sin t \cos t \end{pmatrix} \cdot \begin{pmatrix} -2s^2\cos t \\ -2s^2 \sin t \\ s \end{pmatrix} ds\,dt$$

But this is going to be really messy.

See Method 2.

Method 2 : Add the disk $x^2+y^2 \le 1$ so that we have a closed surface. Then use the divergence thm:

$$\iiint_R 1+0-1 \, dV = \boxed{0}$$

Now we need to subtract off the flux over the disk.

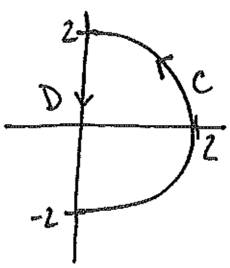
$$C(s,t) = \begin{pmatrix} s\cos t \\ s\sin t \\ 0 \end{pmatrix} \quad s\in[0,1],\ t\in[0,2\pi], \quad C_s \times C_t = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}$$

$$\text{Flux} = \int_0^{2\pi}\int_0^1 \begin{pmatrix} s\cos t \\ 1+\sin(s\cos t) \\ s^2\cos t\sin t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -s \end{pmatrix} ds\,dt$$

$$= \int_0^{2\pi}\int_0^1 -s^3\cos t\sin t \, ds\,dt = \int_0^{2\pi} -\frac{s^4}{4}\cos t\sin t \Big|_0^1 dt$$

$$= -\frac{1}{4}\int_0^{2\pi}\cos t\sin t\,dt = -\frac{1}{8}\sin^2 t\Big|_0^{2\pi} = \boxed{0}$$

So, the total flux over the paraboloid is $0-0=\boxed{0}$

(11) 

Since F is pretty ugly, try to close up the curve with the vertical seg from $(0,2)$ to $(0,-2)$, call it D, & use Stokes' Thm.

$$\nabla \times F = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad C(s,t) = \begin{pmatrix} s\cos t \\ s\sin t \\ 0 \end{pmatrix}, \quad \begin{matrix} s\in[0,2] \\ t\in[-\frac{\pi}{2},\frac{\pi}{2}] \end{matrix}, \quad C_s\times C_t = \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}$$

$$\int_{-\pi/2}^{\pi/2}\int_0^2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} ds\,dt = \int_{-\pi/2}^{\pi/2}\int_0^2 -s\,ds\,dt = \int_{-\pi/2}^{\pi/2} -\frac{s^2}{2}\Big|_0^2 dt$$

$$= \int_{-\pi/2}^{\pi/2} -2\,dt = -2\pi$$

Now, find the work over D:

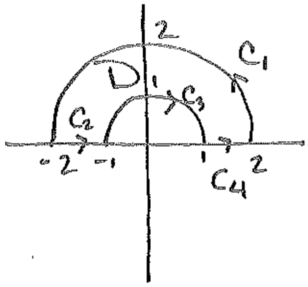
$$\gamma(t) = \begin{pmatrix} 0 \\ -t \\ 0 \end{pmatrix} \quad t\in[-2,2], \quad \gamma'(t) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\int_{-2}^{2} \begin{pmatrix} \pi t \\ \pi \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} dt = \int_{-2}^{2} -\pi\,dt = -4\pi$$

So, $\int_C F\cdot ds = \iint_R \nabla\times F\cdot dr - \int_D F\cdot ds$

$$= -2\pi - (-4\pi) = \boxed{2\pi}$$

(16)



Green's Thm: $\int_C F \cdot ds = \iint_D (3y - 3y^2)\, dA$

$$= \int_0^{\pi} \int_1^{2} (3r\sin\theta - 3r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi} r^3\sin\theta - \tfrac{3}{4} r^4 \sin^2\theta \Big|_1^2 \, d\theta$$

$$= \int_0^{\pi} 8\sin\theta - \sin\theta - 12\sin^2\theta + \tfrac{3}{4}\sin^2\theta \, d\theta$$

$$= \int_0^{\pi} 7\sin\theta - \frac{45}{4}\cdot\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= -7\cos\theta\Big|_0^{\pi} - \frac{45}{8}\theta\Big|_0^{\pi} + \frac{45}{8}\cdot\frac{\sin 2\theta}{2}\Big|_0^{\pi}$$

$$= 7 - (-7) - \frac{45}{8}\pi + 0$$

$$= \boxed{14 - \frac{45}{8}\pi}$$

(17) (a) True (b) True

(18) $C(s,t) = \begin{pmatrix} s\cos t \\ s\sin t \\ 10 + 2s\cos t + 3s\sin t \end{pmatrix}$ $\quad C_s = \begin{pmatrix} \cos t \\ \sin t \\ 2\cos t + 3\sin t \end{pmatrix}$, $s \in [0,1]$, $t \in [0, 2\pi]$

$$C_t = \begin{pmatrix} -s\sin t \\ s\cos t \\ -2s\sin t + 3s\cos t \end{pmatrix}$$

$$C_s \times C_t = \begin{vmatrix} i & j & k \\ \cos t & \sin t & 2\cos t + 3\sin t \\ -s\sin t & s\cos t & -2s\sin t + 3s\cos t \end{vmatrix} = i(-2s\sin^2 t + \cancel{3s\sin t\cos t} - 2s\cos^2 t - 3s\,\cancel{\sin t\cos t}) - j(-2s\,\cancel{\cos t\sin t} + 3s\cos^2 t + 2s\,\cancel{\cos t\sin t} + 3s\sin^2 t) + k(s\cos^2 t + s\sin^2 t)$$

$$= \begin{pmatrix} -2s \\ 3s \\ s \end{pmatrix}.$$

$$\int_0^{2\pi}\int_0^1 \left\| \begin{pmatrix} -2s \\ 3s \\ s \end{pmatrix} \right\| ds\,dt = \int_0^{2\pi}\int_0^1 \sqrt{4s^2 + 9s^2 + s^2}\, ds\,dt = \sqrt{14}\int_0^{2\pi}\int_0^1 s\,ds\,dt$$

$$= \sqrt{14}\int_0^{2\pi} \frac{s^2}{2}\Big|_0^1 dt = \frac{\sqrt{14}}{2}\int_0^{2\pi} dt = \frac{\sqrt{14}}{2}\cdot 2\pi = \boxed{\sqrt{14}\,\pi}$$

(19) Add 2 disks, $x^2+z^2 \le 1$ at either end, $y=0$ & $y=3$. Then by the divergence thm:

$$\int_{\Sigma} F \cdot d\sigma = \iint_D (1+1-1)\, dA = \text{volume of cylinder} = 3\pi.$$

Now find the flux of the 2 disks directly:

$$C_1(r,\theta) = \begin{pmatrix} r\cos\theta \\ 0 \\ r\sin\theta \end{pmatrix} \quad \& \quad C_2(r,\theta) = \begin{pmatrix} r\cos\theta \\ 0 \\ r\sin\theta \end{pmatrix} \qquad \begin{matrix} r \in [0,1] \\ \theta \in [0,2\pi] \end{matrix}$$

$$(C_1)_r = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix} \qquad (C_1)_\theta = \begin{pmatrix} -r\sin\theta \\ 0 \\ r\cos\theta \end{pmatrix}$$

$$(C_1)_r \times (C_1)_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & 0 & \sin\theta \\ -r\sin\theta & 0 & r\cos\theta \end{vmatrix} = -j(r\cos^2\theta + r\sin^2\theta) = \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix}$$

Flux over C_1:

$$\int_0^{2\pi}\int_0^1 \begin{pmatrix} r\cos\theta \\ 0 \\ -r\sin\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix} dr\, d\theta = 0$$

Flux over C_2: (C_2 has same normal, but should point right: $\begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix}$).

$$\int_0^{2\pi}\int_0^1 \begin{pmatrix} r\cos\theta \\ 3 \\ -r\sin\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix} dr\, d\theta = \int_0^{2\pi}\int_0^1 3r\, dr\, d\theta = \int_0^{2\pi} \frac{3r^2}{2}\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{3}{2}\, d\theta = \frac{3}{2} \cdot 2\pi = 3\pi$$

Flux over cylinder = flux over solid surface − flux over disks

$= 3\pi - 0 - 3\pi$

$= \boxed{0}$

[This could also have been done directly]

(20) This is the square $x \in [1,2]$, $y \in [-1,0]$ in the plane $z = 3$.

$C(s,t) = \begin{pmatrix} s \\ t \\ 3 \end{pmatrix}$, $s \in [1,2]$, $t \in [-1,0]$ $\quad C_s = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $C_t = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$C_s \times C_t = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\text{work} = \int_{\Sigma} \nabla \times F \cdot d\sigma = \int_{-1}^{0} \int_{1}^{2} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ds\,dt$$

$$= \int_{-1}^{0} \int_{1}^{2} 3\,ds\,dt = 3(\text{area of square})$$

$$= 3 \cdot 1 = \boxed{3}$$