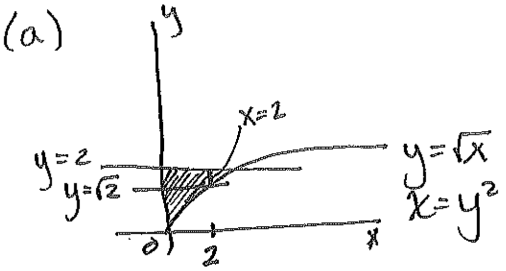
Final Review Part I Solutions

① (a)



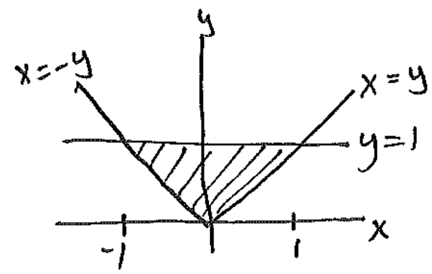
$$\int_0^{\sqrt{2}}\int_0^{y^2} y\,dx\,dy + \int_{\sqrt{2}}^{2}\int_0^{2} y\,dx\,dy$$

$$= \int_0^{\sqrt{2}} xy\Big|_0^{y^2} dy + \int_{\sqrt{2}}^{2} xy\Big|_0^{2} dy$$

$$= \int_0^{\sqrt{2}} y^3\,dy + \int_{\sqrt{2}}^{2} 2y\,dy$$

$$= \frac{y^4}{4}\Big|_0^{\sqrt{2}} + y^2\Big|_{\sqrt{2}}^{2} = 1 + 4 - 2 = \boxed{3}$$

(b)



$$\int_{-1}^{0}\int_{-x}^{1} x^2\,dy\,dx + \int_0^1\int_x^1 x^2\,dy\,dx$$

$$= \int_{-1}^{0} x^2y\Big|_{-x}^{1} dx + \int_0^1 x^2y\Big|_x^1 dx$$

$$= \int_{-1}^{0}(x^2 + x^3)\,dx + \int_0^1 (x^2 - x^3)\,dx$$

$$= \frac{x^3}{3} + \frac{x^4}{4}\Big|_{-1}^{0} + \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{3} - \frac{1}{4} = \frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$$

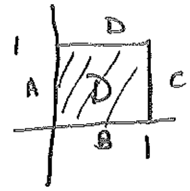
② Cylindrical coordinates:

$$\int_0^{2\pi}\int_0^2\int_0^3 (r^2 - z^2)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_0^3 (r^3 - rz^2)\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3z - \frac{rz^3}{3}\Big|_0^3 dr\,d\theta = \int_0^{2\pi}\int_0^2 (3r^3 - 9r)\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{3}{4}r^4 - \frac{9}{2}r^2\Big|_0^2 d\theta = \int_0^{2\pi} 12 - 18\,d\theta = -6\int_0^{2\pi} d\theta = \boxed{-12\pi}$$

③ $f(x,y) = 3xy - x^2 - y \ , \ 0 \le x \le 1, \ 0 \le y \le 1.$



Interior:

$$f_x = 3y - 2x = 0$$
$$f_y = 3x - 1 = 0 \longrightarrow x = 1/3$$
$$3y - 2(1/3) = 0$$
$$3y = 2/3$$
$$y = 2/9$$

(x,y)	$f(x,y)$
$(1/3, 2/9)$	$\frac{2}{9} - \frac{1}{9} - \frac{2}{9} = -\frac{1}{9}$
$(0,0)$	0
$(2/3, 1)$	$2 - 4/9 - 1 = 5/9$
Corners $(0,1)$	-1
$(1,0)$	-1
$(1,1)$	1

$$\boxed{\begin{array}{l}\text{Max of } 1 \text{ at } (1,1)\\ \text{Min of } -1 \text{ at } (0,1) \ \& \ (1,0)\end{array}}$$

Boundary: (Substitution)

A: $x = 0$

$f(0,y) = -y$

$f'(0,y) = -1 \neq 0$ (no crit. pts)

B: $y = 0$

$f(x,0) = -x^2$

$f'(x,0) = -2x = 0$

$x = 0$ $\quad (0,0)$

C: $x = 1$

$f(1,y) = 3y - 1 - y$

$f'(1,y) = 3 - 1 = 2 \neq 0$ (no crit. pts)

D: $y = 1$

$f(x,1) = 3x - x^2$

$f'(x,1) = 3 - 2x = 0$

$x = 2/3$

④ $xy - yz + e^{xz} = 3$. Let $f(x,y,z) = xy - yz + e^{xz}$. Then the tan. plane to $f(x,y,z) = 3$ at $(0,-1,2)$ is $\nabla f \cdot \begin{pmatrix} x \\ y+1 \\ z-2 \end{pmatrix} = 0$:

$$\nabla f\Big|_{(0,-1,2)} = \begin{pmatrix} y + ze^{xz} \\ x - z \\ -y + xe^{xz} \end{pmatrix}\Bigg|_{(0,-1,2)} = \begin{pmatrix} -1 + 2e^0 \\ 0 - 2 \\ 1 + 0e^0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

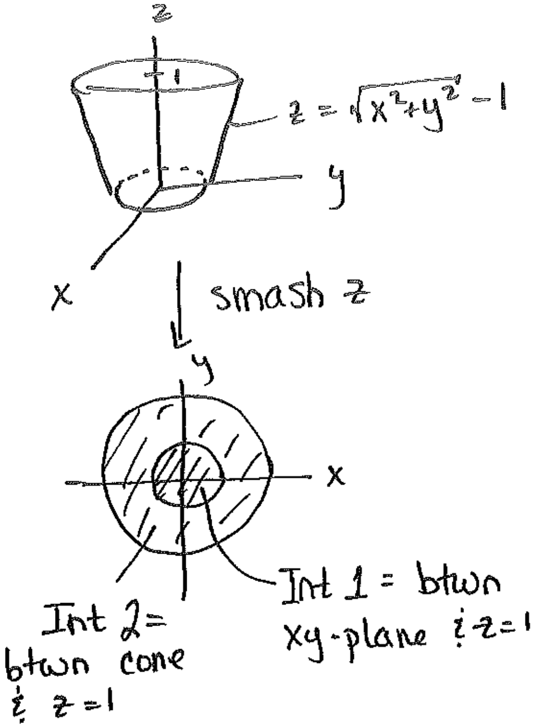
$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y+1 \\ z-2 \end{pmatrix} = 0$$

$$x - 2(y+1) + (z-2) = 0$$

Approx: $0.1 - 2(-.1) + z - 2 = 0$

$$.1 + .2 + z - 2 = 0$$
$$z \approx 2 - .1 - .2$$
$$\boxed{z \approx 1.7}$$

⑤



Cylindrical

$$\int_0^{2\pi}\int_0^1\int_0^1 \frac{r}{z+1}\,dz\,dr\,d\theta \;+$$

$$\int_0^{2\pi}\int_1^2\int_{r-1}^1 \frac{r}{z+1}\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\ln(z+1)\Big|_0^1 dr\,d\theta \;+$$

$$\int_0^{2\pi}\int_1^2 r\ln(z+1)\Big|_{r-1}^1 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\ln 2\,dr\,d\theta + \int_0^{2\pi}\int_1^2 [r\ln 2 - r\ln(r)]\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2}\cdot\ln 2\Big|_0^1 d\theta + \int_0^{2\pi}\frac{r^2}{2}\ln 2\Big|_1^2 d\theta - \underbrace{\int_0^{2\pi}\int_1^2 r\ln r\,dr\,d\theta}$$

$u = \ln r \quad v = \frac{1}{2}r^2$

$du = \frac{dr}{r} \quad dv = r\,dr$

$$= \frac{1}{2}r^2\ln r\Big|_1^2 - \int_1^2 \frac{1}{2}r^2\cdot\frac{1}{r}dr$$

$$= 2\ln 2 - 0 - \frac{1}{4}r^2\Big|_1^2$$

$$= 2\ln 2 - 1 + 1/4$$

$$= \cancel{\int_0^{2\pi}\frac{1}{2}\ln 2\,d\theta} + \cancel{\int_0^{2\pi} 2\ln 2 - \frac{1}{2}\ln 2\,d\theta} - \cancel{\int_0^{2\pi} 2\ln 2} - 3/4\,d\theta$$

$$= \int_0^{2\pi} + 3/4\,d\theta = \frac{3}{4}\cdot 2\pi = \boxed{\frac{3\pi}{2}}$$

⑥ $\rho = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial\rho}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\boxed{\frac{\partial\rho}{\partial x} = \frac{x}{\rho}}$$

$$\nabla\cdot F = \frac{\partial(\rho^2 x)}{\partial x} + \frac{\partial(\rho^2 y)}{\partial y} + \frac{\partial(\rho^2 z)}{\partial z}$$

$$= 2\rho\cdot\frac{\partial\rho}{\partial x}\cdot x + \rho^2 + 2\rho\frac{\partial\rho}{\partial y}y + \rho^2 + 2\rho\frac{\partial\rho}{\partial z}z + \rho^2$$

$$= 2\cancel{\rho}\cdot\frac{x}{\cancel{\rho}}\cdot x + 3\rho^2 + 2\cancel{\rho}\cdot\frac{y}{\cancel{\rho}}\cdot y + 2\cancel{\rho}\frac{z}{\cancel{\rho}}\cdot z$$

$$= 2x^2 + 2y^2 + 2z^2 + 3\rho^2$$

$$= 2\rho^2 + 3\rho^2 = \boxed{5\rho^2}$$

⑦ $f(x,y) = xe^{x-2y}$. Approx. $f(1.99, 1.02)$

Tan. plane to f at (2,1) is given by:

$$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$$

$$f(2,1) = 2e^{2-2} = 2e^0 = 2$$

$$f_x\Big|_{(2,1)} = e^{x-2y} + xe^{x-2y}\Big|_{(2,1)} = e^0 + 2e^0 = 1+2 = 3$$

$$f_y(2,1) = -2xe^{x-2y}\Big|_{(2,1)} = -2 \cdot 2e^0 = -4$$

$$z = 2 + 3(x-2) - 4(y-1).$$

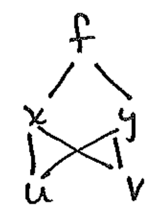
$$z \approx 2 + 3(1.99-2) - 4(1.02-1)$$

$$\approx 2 + 3(-0.01) - 4(0.02)$$

$$\approx 2 - 0.03 - 0.08$$

$$\approx 2 - 0.11$$

$$\boxed{z \approx 1.89}$$



⑧ $g(u,v) = f(u\ln v,\ u+v)$

(a)

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\boxed{\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \cdot \ln v + \frac{\partial f}{\partial y} \cdot 1}$$

(b) $$\frac{\partial^2 g}{\partial u \partial v} = \frac{\partial^2 g}{\partial v \partial u} = \frac{\partial}{\partial v}\Big(\underbrace{\frac{\partial f}{\partial x} \ln v}_{\text{product rule}} + \frac{\partial f}{\partial y}\Big) = \left[\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{\partial y}{\partial v}\right] \ln v +$$

$$\frac{\partial f}{\partial x} \cdot \frac{1}{v} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial y}{\partial v}$$

$$\boxed{= \left(\frac{\partial^2 f}{\partial x^2} \cdot \frac{u}{v} + \frac{\partial^2 f}{\partial y \partial x} \cdot 1\right) \ln v + \frac{\partial f}{\partial x} \cdot \frac{1}{v} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{u}{v} + \frac{\partial^2 f}{\partial y^2} \cdot 1}$$

⑨ $f(x,y) = x^2 - 3y^2 + y^3$ C.P.'s: $(0,0)$, $(0,2)$

$f_x = 2x = 0 \to x = 0$

$f_y = -6y + 3y^2 = 0$

$3y(-2+y) = 0$

$y = 0$ or $y = 2$

2^{nd} derivative test:	$(0,0)$	$(0,2)$
$f_{xx} = 2$	2	2
$f_{yy} = -6 + 6y$	-6	6
$f_{xy} = 0$	0	0
$D = f_{xx} f_{yy} - (f_{xy})^2$	$-12 < 0$ saddle	$12 > 0$ $f_{xx} = 2 > 0$ min.

$$\boxed{\begin{array}{l}(0,0) \text{ is a saddle pt} \\ (0,2) \text{ is a local min}\end{array}}$$

⑩ $f(x,y) = xy^2$, $g(x,y) = x^2 + y^2 = 3$ (Lagrange multipliers)

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 3 \end{cases} \quad \text{or} \quad \begin{cases} \nabla g = 0 \\ g(x,y) = 3 \end{cases} \Rightarrow \begin{cases} 2x = 0 \\ 2y = 0 \\ x^2 + y^2 = 3 \end{cases} \Rightarrow \text{no solution}$$

$$\begin{cases} y^2 = \lambda \cdot 2x \\ 2xy = \lambda \cdot 2y \to \lambda = x \text{ or } y = 0 \\ x^2 + y^2 = 3 \end{cases}$$

<u>If $y = 0$:</u> $x = \pm\sqrt{3}$

<u>If $\lambda = x$:</u> $y^2 = 2x^2$

$y = \pm\sqrt{2}\,x$

$x^2 + 2x^2 = 3$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

$y = \pm\sqrt{2}$

(x,y)	$f(x,y)$
$(\sqrt{3}, 0)$	0
$(-\sqrt{3}, 0)$	0
$(1, \pm\sqrt{2})$	2
$(-1, \pm\sqrt{2})$	-2

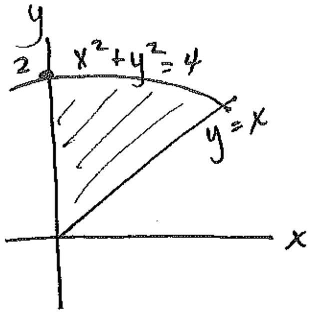
$$\boxed{\begin{array}{l}\text{Maximum of } 2 \text{ at } (1, \pm\sqrt{2}) \\ \text{Minimum of } -2 \text{ at } (-1, \pm\sqrt{2})\end{array}}$$

(11) $\displaystyle\int_3^6 \int_0^2 \frac{1}{(x+y)^2}\, dy\,dx = \int_3^6 \frac{-1}{x+y}\Big|_0^2 dx = \int_3^6 \left(\frac{-1}{x+2} + \frac{1}{x}\right)dx$

$= -\ln(x+2) + \ln(x)\Big|_3^6 = -\ln(8) + \ln(6) - (-\ln(5) + \ln 3)$

$= \boxed{-\ln 8 + \ln 6 + \ln 5 - \ln 3}$

(12)

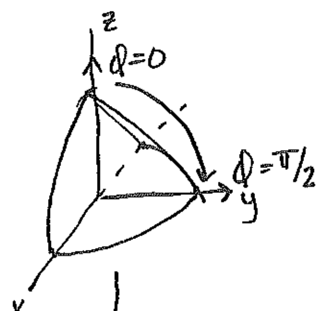


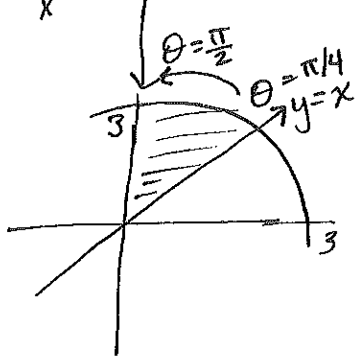
$\displaystyle\int_{\pi/4}^{\pi/2} \int_0^2 \int_0^{r^2} r\,dz\,dr\,d\theta$

$\displaystyle= \int_{\pi/4}^{\pi/2} \int_0^2 rz\Big|_0^{r^2} dr\,d\theta = \int_{\pi/4}^{\pi/2} \int_0^2 r^3\,dr\,d\theta$

$\displaystyle= \int_{\pi/4}^{\pi/2} \frac{r^4}{4}\Big|_0^2 d\theta = \int_{\pi/4}^{\pi/2} 4\,d\theta = 4\theta\Big|_{\pi/4}^{\pi/2} = 4(\pi/2 - \pi/4) = 4\left(\tfrac{\pi}{4}\right) = \boxed{\pi}$

(13)





$\displaystyle\int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho\sin\phi\cos\theta \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$\displaystyle= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^3 \sin^2\phi \cos\theta\, d\rho\, d\phi\, d\theta$

Note: $z \geq 0 \Rightarrow 0 \leq \phi \leq \pi/2$

$x \geq 0$ & $y \geq x \Rightarrow \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

sphere $\Rightarrow 0 \leq \rho \leq 3$

(14) Spherical coordinates: $z \geq 0 \Rightarrow 0 \leq \phi \leq \pi/2$

$\displaystyle\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^2 \frac{1}{\rho} \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho \sin\phi\, d\rho\, d\phi\, d\theta$

$\displaystyle= \int_0^{2\pi} \int_0^{\pi/2} \frac{\rho^2}{2} \sin\phi \Big|_{\rho=1}^{\rho=2} d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/2} \left(2\sin\phi - \tfrac{1}{2}\sin\phi\right) d\phi\, d\theta$

$\displaystyle= \int_0^{2\pi} -\frac{3}{2}\cos\phi \Big|_0^{\pi/2} d\theta = \int_0^{2\pi} \frac{3}{2}\, d\theta = \frac{3}{2} \cdot 2\pi = \boxed{3\pi}$

(15) Parametrize S: $C(s,t) = (s, t, 1-t^2)$, $(s,t) \in [0,1] \times [0,1]$

$$C_s = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad C_t = \begin{pmatrix} 0 \\ 1 \\ -2t \end{pmatrix}$$

$$C_s \times C_t = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -2t \end{vmatrix} = i(0) - j(-2t) + k(1) = \begin{pmatrix} 0 \\ 2t \\ 1 \end{pmatrix}$$

$$\|C_s \times C_t\| = \sqrt{4t^2+1}$$

$$\int_S xy \, d\sigma = \int_0^1 \int_0^1 (s \cdot t) \cdot \|C_s \times C_t\| \, ds dt = \int_0^1 \int_0^1 st\sqrt{4t^2+1} \, ds dt$$

$$= \int_0^1 \frac{s^2}{2} \cdot t\sqrt{4t^2+1} \Big|_{s=0}^{s=1} dt = \int_0^1 \frac{1}{2} t \sqrt{4t^2+1} \, dt \qquad \begin{matrix} u = 4t^2+1 \\ du = 8t\,dt \end{matrix}$$

$$= \frac{1}{16} \int_1^5 u^{1/2} du = \frac{1}{16} \cdot \frac{2}{3} u^{3/2} \Big|_1^5 = \boxed{\frac{1}{24}(5\sqrt{5} - 1)}$$

(16) $\gamma(t) = \begin{pmatrix} e^t \cos t \\ e^t \sin t \\ e^t \end{pmatrix}$, $t \in [-1,1]$. $\quad \gamma'(t) = \begin{pmatrix} e^t \cos t - e^t \sin t \\ e^t \sin t + e^t \cos t \\ e^t \end{pmatrix}$

length of $C = \int_\gamma ds = \int_{-1}^1 \|\gamma'(t)\| dt$

$$\|\gamma'(t)\| = \sqrt{e^{2t}\cos^2 t - \cancel{2e^t \cos t \sin t} + e^{2t}\sin^2 t + e^{2t}\sin^2 t + \cancel{2e^t \sin t \cos t} + e^{2t}\cos^2 t + e^{2t}}$$

$$= \sqrt{e^{2t}(\cos^2 t + \sin^2 t) + e^{2t}(\cos^2 t + \sin^2 t) + e^{2t}} = \sqrt{3e^{2t}} = \sqrt{3}e^t$$

length of $C = \int_{-1}^1 \sqrt{3} e^t dt = \sqrt{3}e^t \Big|_{-1}^1 = \boxed{\sqrt{3}e - \sqrt{3}e^{-1}}$

$$T = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \boxed{\begin{pmatrix} (\cos t - \sin t)/\sqrt{3} \\ (\sin t + \cos t)/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = T}$$

$$N = \frac{T'(t)}{\|T'(t)\|} : \quad T'(t) = \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \\ 0 \end{pmatrix}$$

$$\|T'(t)\| = \frac{1}{\sqrt{3}} \cdot \sqrt{\sin^2 t + 2\cos t \sin t + \cos^2 t + \cos^2 t - 2\cos t \sin t + \sin^2 t}$$

$$= \frac{1}{\sqrt{3}} \sqrt{1+1} = \sqrt{\frac{2}{3}}$$

$$N = \frac{T'}{\|T'\|} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} -\sin t - \cos t \\ \cos t - \sin t \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} (-\sin t - \cos t)/\sqrt{2} \\ (\cos t - \sin t)/\sqrt{2} \\ 0 \end{pmatrix} = N}$$

$$\text{Acceleration} = \gamma''(t) = \begin{pmatrix} e^t\cos t - e^t \sin t - (e^t \sin t + e^t \cos t) \\ e^t \sin t + e^t \cos t + (e^t \cos t - e^t \sin t) \\ e^t \end{pmatrix}$$

$$\boxed{\gamma''(t) = \begin{pmatrix} -2e^t \sin t \\ 2e^t \cos t \\ e^t \end{pmatrix}}$$

$$\text{Curvature} = \kappa = \frac{\|T'(t)\|}{\|\gamma'(t)\|} = \frac{\sqrt{2/3}}{\sqrt{3}\, e^t} = \boxed{\frac{\sqrt{2}}{3} e^{-t} = \kappa}$$

(17) $g(x,y,z) = x^2 + 1 - 2yz = 0$ (constraint)

minimize $f(x,y,z) = x^2+y^2+z^2$ [the square of distance to $(0,0,0)$]

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases} \quad \text{or} \quad \begin{cases} \nabla g = 0 \\ g(x,y,z) = 0 \end{cases} \Rightarrow \begin{cases} 2x = 0 \\ -2z = 0 \\ -2y = 0 \\ x^2+1-2yz=0 \end{cases} \Rightarrow \text{no solution.}$$

$$\begin{cases} 2x = \lambda \cdot 2x \\ 2y = \lambda \cdot -2z \\ 2z = \lambda \cdot -2y \\ x^2+1-2yz=0 \end{cases}$$

$2x = \lambda \cdot 2x \longrightarrow \lambda = 1$ or $x = 0$

<u>If $x=0$</u>: $2yz = 1$ (from last eqn)

$$z = \frac{1}{2y}$$

$$\begin{cases} 2y = \lambda \cdot -\cancel{2}\left(\frac{1}{\cancel{2}y}\right) \\ 2\left(\frac{1}{\cancel{2}y}\right) = \lambda \cdot -2y \end{cases} \longrightarrow 2y^2 = -\lambda \longrightarrow \lambda = -2y^2$$

$1 = (-2y^2)(-2y^2)$

$1 = 4y^4$

$y^4 = 1/4$

$y = \pm \frac{1}{\sqrt{2}} \rightarrow$ If $y = \frac{1}{\sqrt{2}}$, $z = \frac{1}{2} \cdot \frac{\sqrt{2}}{1} = \sqrt{2}/2$

If $y = -\frac{1}{\sqrt{2}}$, $z = -\frac{\sqrt{2}}{2}$

<u>If $\lambda = 1$</u>:

$2y = -2z \to y = -z$

$2z = -2y$

$x^2 + 1 + 2z^2 = 0$

$$\underbrace{x^2 + 2z^2}_{\text{positive}} = \underbrace{-1}_{\text{negative}} \ ↯$$

(x,y,z)	$f(x,y,z)$
$(0,\frac{1}{\sqrt2},\sqrt2/2)$	$0^2+\frac12+\frac24=1$
$(0,-\frac{1}{\sqrt2},-\frac{\sqrt2}{2})$	$0^2+\frac12+\frac24=1$

Minimum dist $=1$ & 2 closest pts are $(0,\frac{1}{\sqrt2},\frac{\sqrt2}{2})$ & $(0,-\frac{1}{\sqrt2},-\frac{\sqrt2}{2})$

(18) $x-y+e^{xz-x}+z=2$, near $(1,1,1)$

(a) If $g(x,y,z)=x-y+e^{xz-x}+z$, then the tangent plane is given by $\nabla g\cdot\begin{pmatrix}x-1\\y-1\\z-1\end{pmatrix}=0$

$$\nabla g=\left.\begin{pmatrix}1+(z-1)e^{xz-x}\\-1\\xe^{xz-x}+1\end{pmatrix}\right|_{(1,1,1)}=\begin{pmatrix}1\\-1\\1e^{1-1}+1\end{pmatrix}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

So the tangent plane is:

$$\boxed{0=(x-1)+-1(y-1)+2(z-1).}$$

(b) $f(x,y)$ decreases most rapidly in the direction of $-\nabla f$
Since $\nabla f=\begin{pmatrix}\partial f/\partial x\\\partial f/\partial y\end{pmatrix}$ & $z=f(x,y)$, then we need to use implicit differentiation to find $\partial z/\partial x$ & $\partial z/\partial y$.

$\frac{\partial z}{\partial x}$: $1+(z+x\frac{\partial z}{\partial x}-1)e^{xz-x}+\frac{\partial z}{\partial x}=0$. near $(1,1)$, so plug in $x=1, y=1$ (& so $z=1$, as in (a))

$$1+(\cancel{1}+\frac{\partial z}{\partial x}\cancel{-1})e^{1-1}+\frac{\partial z}{\partial x}=0$$

$$2\frac{\partial z}{\partial x}=-1$$

$$\partial z/\partial x=-1/2$$

$\frac{\partial z}{\partial y}$: $-1+x\frac{\partial z}{\partial y}e^{xz-x}+\frac{\partial z}{\partial y}=0$. Again, plug in $(1,1,1)$:

$$-1+\frac{\partial z}{\partial y}e^{1-1}+\frac{\partial z}{\partial y}=0$$

$$2\frac{\partial z}{\partial y}=1$$

$$\frac{\partial z}{\partial y}=1/2$$

So direction is $-\begin{pmatrix}-1/2\\1/2\end{pmatrix}=\boxed{\begin{pmatrix}1/2\\-1/2\end{pmatrix}}$

(b) (continued)

OR: Take the tangent plane from (a) & rewrite in form $z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$ by solving for z:

$$2(z-1) = -(x-1) + (y-1)$$

$$z - 1 = -\frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

$$z = 1 \underbrace{-\frac{1}{2}}_{f_x(1,1)}(x-1) \underbrace{+\frac{1}{2}}_{f_y(1,1)}(y-1)$$

So $\nabla f(1,1) = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$ & thus the direction is $\boxed{\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}}$

(c) Solve $\nabla f(1,1) \cdot \vec{v} = 1/4$, where $\vec{v}$ is a unit vector

$$\begin{cases} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4} \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} -\frac{1}{2}x + \frac{1}{2}y = \frac{1}{4} \\ x^2 + y^2 = 1 \end{cases}$$

because $\|\vec{v}\| = 1$

$$-2x + 2y = 1 \rightarrow 2y = 1 + 2x$$

$$y = \frac{1}{2} + x$$

$$x^2 + \left(\frac{1}{2} + x\right)^2 = 1$$

$$x^2 + \frac{1}{4} + x + x^2 = 1$$

$$2x^2 + x - \frac{3}{4} = 0$$

$$8x^2 + 4x - 3 = 0$$

$$\& \quad x = \frac{-4 \pm \sqrt{16 + 4 \cdot 8 \cdot 3}}{16} = \frac{-4 \pm \sqrt{16(1+6)}}{16} = \frac{-4 \pm 4\sqrt{7}}{16}$$

$$x = \frac{-1 \pm \sqrt{7}}{4}, \quad y = \frac{1}{2} + \frac{-1 \pm \sqrt{7}}{4}$$

Yes, there is such a direction.

(19) $z = x^2 - 2xy - y^2 - 8x + 4y = f(x,y)$

Tan. plane at (a,b) is: $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

This is a horizontal plane when it is of the form $z = \text{constant}$, that is, where $f_x(a,b)$ & $f_y(a,b) = 0$. So we need to solve $\nabla f = 0$.

$f_x = 2x - 2y - 8 = 0 \longrightarrow 2x = 2y + 8$

$f_y = -2x - 2y + 4 = 0 \qquad x = y + 4$

$-2(y+4) - 2y + 4 = 0$

$-2y - 8 - 2y + 4 = 0$

$-4y - 4 = 0$

$y = -1 \longrightarrow x = -1 + 4$

$x = 3$

The tan. plane is horizontal at $\boxed{(3,-1).}$