1. Suppose A is an $m \times n$ matrix such that $\text{Nul}(A) = \{\vec{0}\}$. Prove that $A^T A$ is invertible.

2. Find, with proof, all possible real eigenvalues of an orthogonal matrix.

3. Let $L\colon V \to V$ be a linear map on a vector space V and $\vec{v} \in V$ a vector with the property that $L^{k-1}(\vec{v}) \neq 0$ but $L^k(\vec{v}) = 0$. Show that $\vec{v}, L(\vec{v}), \ldots, L^{k-1}(\vec{v})$ are linearly independent.

4. Are there square matrices A, B with the property that $AB - BA = I$? Either give an example of such matrices or prove that no such matrices exist.

5. Suppose A and B are $n \times n$ matrices that commute, and suppose that B has n distinct eigenvalues.

   (a) Show that if $B\vec{v} = \lambda\vec{v}$, then $BA\vec{v} = \lambda A\vec{v}$.

   (b) Show that every eigenvector for B is also an eigenvector for A.

   (c) Show that A is diagonalizable.

   (d) Show that AB is diagonalizable.

6. Prove or disprove:

   (a) If V is a finite dimensional vector space and $T\colon V \to V$ is a linear transformation, then $\ker(T) = \{\vec{0}\}$ if and only if $\text{Im}(T) = V$. Is this true if V is infinite dimensional?

   (b) $\det(A + B) = \det(A) + \det(B)$ for square matrices A and B of the same size.

   (c) Let n be an odd number. If an $n \times n$ matrix $A = (a_{ij})$ satisfies $a_{ij} = -a_{ji}$ for all $i, j = 1, \ldots, n$, then $\det(A) = 0$.

7. Given an example of:

   (a) A non-diagonalizable matrix with real entries.

   (b) Two orthonormal bases of $\mathbb{R}^3$.

   (c) A linear transformation T that is not the identity transformation with $\ker(T) = \{\vec{0}\}$.

   (d) A non-identity matrix A for which every non-zero vector is an eigenvector.

8. Consider

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 3 \\ 5 \\ 7 \\ -3 \end{pmatrix}.$$

   (a) Find a QR factorization of A.

   (b) Use the QR factorization you found in part (a) to find the least squares solution to

$$A\vec{x} = \vec{b}.$$

9. Define $T\colon \mathbb{P}_2 \to \mathbb{R}^3$ by

$$T(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}.$$

   (a) Find the image under T of $p(t) = 3 + 5t$.

   (b) Show that T is a linear transformation.

   (c) Find the kernel of T. Does this imply that T is injective? Surjective?

   (d) Let $\mathcal{B} = \{1, t, t^2\}$ be a basis for $\mathbb{P}_2$. Find $[T]_{\mathcal{B}}$, the matrix of T with respect to the basis $\mathcal{B}$.

10. Let A be an $m \times n$ matrix and suppose there is a matrix C such that $AC = I_m$. Show that the equation $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

11. Show that if λ is an eigenvalue of A, then it is an eigenvalue of A^T.

12. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1-2t+t^2, 3-5t+4t^2, 2t+3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then find the $\mathcal{B}$–coordinate vector for $p(t) = -1 + 2t$.

13. Find a basis for the row space, the column space, and the null space of A, where

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{pmatrix}.$$

14. Solve the following differential equations.

   (a) $y'' + 4y' + 4y = 0$, $y(-1) = 2$, $y'(-1) = 1$.

   (b) $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 2$.

   (c) $y'' + 2y' + 5y = 4e^{-t}\cos(2t)$, $y(0) = 1$, $y'(0) = 0$.

   (d) $y'' - 2y' + y = \frac{e^t}{1+t^2}$.

(e) $y^{(4)} + 2y'' + y = 0.$

(f) $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$

15. Find the solution to the heat conduction problem

$$\begin{aligned} 100u_{xx} &= u_t, \qquad 0 < x < 1, \quad t > 0; \\ u(0,t) &= 0, \qquad u(1,t) = 0, \quad t > 0; \\ u(x,0) &= \sin 2\pi x - \sin 5\pi x, \qquad 0 \le x \le 1. \end{aligned}$$

16. Determine if the method of separation of variables can be used to replace the partial differential equation

$$xu_{xx} + u_t = 0$$

by a pair of ordinary differential equations. If so, find (but do not solve) the pair of equations.

17. Find the Fourier cosine series for

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

with period 4.

18. Consider an elastic string of length L whose ends are held fixed. The string is set in motion with no initial velocity from an initial position

$$u(x,0) = \begin{cases} \frac{2x}{L}, & 0 \le x \le L/2 \\ \frac{2(L-x)}{L}, & L/2 < x \le L \end{cases}.$$

Find the displacement $u(x,t)$. (Note: your answer should contain a and L.)