1. Let V be a vector space with inner product $\langle \cdot, \cdot \rangle$. Prove the *parallelogram identity*: for every $\vec{v}, \vec{w} \in V$,

$$\|\vec{v} + \vec{w}\|^2 + \|\vec{v} - \vec{w}\|^2 = 2\|\vec{v}\|^2 + 2\|\vec{w}\|^2.$$

   Give a geometric interpretation of this identity for $V = \mathbb{R}^n$.

2. Let $V = \mathcal{C}([0, 1])$, and let $w \in V$. For $p, q \in V$, define $\langle p, q \rangle = \int_0^1 p(x)q(x)w(x)\,dx$. Suppose that $w(a) > 0$ for all $a \in [0, 1]$. Does it follow that the above defines an inner product on V? Justify your assertion.

3. Let $S \subset \mathbb{R}^4$ be the subspace spanned by the two vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and let T be the orthogonal complement of S. Find an orthogonal basis for T.

4. Let $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)\,dx$ be an inner product, and let $\mathcal{B} = \{1, x, 3x^2 - 1\}$ be an orthogonal basis for $\mathbb{P}_2$. Let $\mathcal{S} = \operatorname{span}(x, x^2) \subset \mathbb{P}_2$. Using the basis $\mathcal{B}$, find the linear map $P \colon \mathbb{P}_2 \to \mathbb{P}_2$ that is the orthogonal projection from $\mathbb{P}_2$ onto $\mathcal{S}$.

5. Let $\vec{v}_1, \ldots, \vec{v}_k$ be vectors in a vector space with inner product $\langle \cdot, \cdot \rangle$. Define the *Gram determinant* by $G(\vec{v}_1, \ldots, \vec{v}_k) = \det(\langle \vec{v}_i, \vec{v}_j \rangle)$, i.e., the determinant of the matrix with entry $\langle \vec{v}_i, \vec{v}_j \rangle$ in the i, j–th position.

   (a) If $\vec{v}_1, \ldots, \vec{v}_k$ are orthogonal, compute their Gram determinant.

   (b) Show that $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent if and only if their Gram determinant is not zero.

6. Suppose that A is a real $n \times n$ symmetric matrix with two equal eigenvalues. If $\vec{v}$ is any vector, show that the vectors $\vec{v}, A\vec{v}, \ldots, A^{n-1}\vec{v}$ are linearly dependent.

7. Let $\vec{u} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, and $W = \operatorname{Span}\{\vec{u}, \vec{v}\}$.

   (a) Find an orthogonal basis for W.

   (b) Find the closest point in W to $\vec{y}$.

(c) What is the shortest distance between $\vec{y}$ and W.

8. Suppose a matrix A is similar to $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. For each, give a proof or a counterexample.

   (a) $A^2 = A$.

   (b) $\det A = 0$

   (c) $\operatorname{tr} A = 1$

   (d) $\lambda = 0$ is an eigenvalue of A

   (e) $\lambda = 1$ is an eigenvalue of A

   (f) $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of A.

9. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

   Find an orthogonal matrix P and a diagonal matrix D such that $D = S^{-1}AS$.

10. Is the set of all vectors $\vec{v} \in \mathbb{R}^n$ with $\|\vec{v}\| = 1$ a subspace of $\mathbb{R}^n$? Why or why not?

11. Let $\vec{v} \in \mathbb{R}^n$ be a unit vector, and let $Q = I - 2\vec{v}\vec{v}^T$. Show that $Q^2 = I$.

12. Find the general solution to

    (a) $3y'' + 2y' + y = t^2$.

    (b) $y'' + 3y' + 2y = e^{-t}$.

13. Solve the initial value problem

$$y'' - 2y' - 8y = te^{4t}, \qquad y(0) = 0, \qquad y'(0) = 5.$$

14. Find all solutions of the differential equation

$$y^{(iv)} + 4y'' = 0.$$