Solutions to Monday's (4/6) Problems

④ $f(x,y) = x^3 \cdot y^5$ $\qquad g(x,y) = x+y = 8$

$\nabla f = \begin{pmatrix} 3x^2y^5 \\ 5x^3y^4 \end{pmatrix} \qquad \nabla g = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{cases} 3x^2y^5 = \lambda \\ 5x^3y^4 = \lambda \\ ③\ x+y = 8 \end{cases} \quad > \quad \begin{aligned} 3x^2y^5 &= 5x^3y^4 \\ 5x^3y^4 - 3x^2y^5 &= 0 \\ x^2y^4(5x-3y) &= 0 \end{aligned}$$

$x=0$ or $y=0$ or $5x = 3y$

$x = \frac{3y}{5}$

If $x=0$: From ③, $y=8$

If $y=0$: From ③, $x=8$

If $x = \frac{3y}{5}$: From ③, $\frac{3y}{5} + y = 8$

$\frac{8y}{5} = 8$

$y = 5$, so $x = \frac{3 \cdot 5}{5}$

$x = 3$

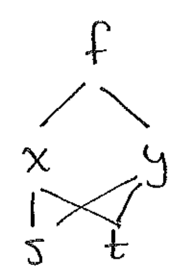
(x,y)	$f(x,y)$
$(0,8)$	0
$(8,0)$	0
$(3,5)$	$3^3 \cdot 5^5 > 0$

Global min of 0 at $(0,8)$ & $(8,0)$. Global max of $3^3 \cdot 5^5$ at $(3,5)$.

① $f(x,y) = xye^{2x}$ $\qquad x = s+t \qquad y = 2s^2+4t^2$

$f_x = ye^{2x} + 2xye^{2x} \qquad x_s = 1 \qquad y_s = 4s$

$f_y = xe^{2x} \qquad x_t = 1 \qquad y_t = 8t$



when $(s,t) = (1,1)$, $(x,y) = (2,6)$, so

$f_x(2,6) = 6e^4 + 24e^4 = 30e^4 \qquad y_s(1,1) = 4$

$f_y = 2e^4 \qquad y_t(1,1) = 8$

So, $f_s = f_x \cdot x_s + f_y \cdot y_s$ & $f_t = f_x \cdot x_t + f_y \cdot y_t$

$= 30e^4 \cdot 1 + 2e^4 \cdot 4 \qquad = 30e^4 \cdot 1 + 2e^4 \cdot 8$

$\boxed{f_s = 38e^4} \qquad \boxed{f_t = 46e^4}$

② • The first part is a directional derivative. $D_{\hat{u}} f = \nabla f \cdot \hat{u}$, where $\hat{u}$ is a unit vector.

If $u = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\hat{u} = \frac{u}{\|u\|} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

$f(x,y) = e^{xy}$, so $\nabla f|_{(1,2)} = \begin{pmatrix} y e^{xy} \\ x e^{xy} \end{pmatrix}\Big|_{(1,2)} = \begin{pmatrix} 2e^2 \\ e^2 \end{pmatrix}$

$$D_{\hat{u}} f = \begin{pmatrix} 2e^2 \\ e^2 \end{pmatrix} \cdot \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \frac{-4e^2}{\sqrt{5}} + \frac{e^2}{\sqrt{5}} = -\frac{3e^2}{\sqrt{5}}.$$

So if you take one step in the direction $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, the surface decreases by $\frac{-3e^2}{\sqrt{5}}$.

- The gradient is always the direction of maximal increase: $\nabla f(1,2) = \begin{pmatrix} 2e^2 \\ e^2 \end{pmatrix}$
- $-\nabla f(1,2) = \begin{pmatrix} -2e^2 \\ -e^2 \end{pmatrix}$ is the direction of minimal increase.
- For no change, you want to be moving along a level set, which is perpendicular to the gradient:

$\begin{pmatrix} 2e^2 \\ e^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0$. There are many answers. Here are 2: $\begin{pmatrix} -e^2 \\ 2e^2 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

③ $f(x,y) = x^3 + x^2 y - y^2 - 4y$. For local extrema, we need to use the 1st & 2nd derivative test.

$$\nabla f = \begin{pmatrix} 3x^2 + 2xy \\ x^2 - 2y - 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$3x^2 + 2xy = 0$

$x^2 - 2y - 4 = 0 \rightarrow y = \frac{x^2 - 4}{2}$

$3x^2 + 2x\left(\frac{x^2-4}{2}\right) = 0$

$3x^2 + x^3 - 4x = 0$

$x(x^2 + 3x - 4) = 0$

$x(x+4)(x-1) = 0$

$x = 0$ or $x = -4$ or $x = +1$

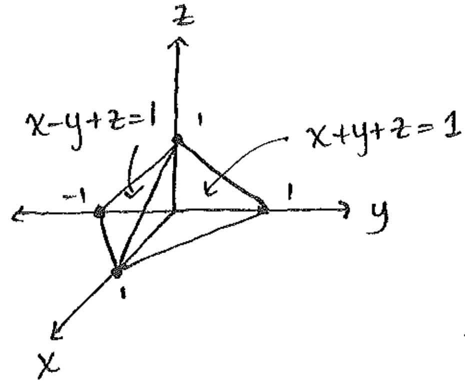
$y = -2$ $\quad y = 6$ $\quad y = -3/2$

2^{nd} deriv. test:

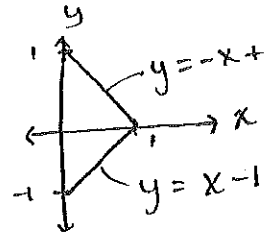
	(x,y)	$(0,-2)$	$(-4,6)$	$(1,-3/2)$
f_{xx}	$6x+2y$	-4	-12	3
f_{xy}	$2x$	0	-8	2
f_{yy}	-2	-2	-2	-2
$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$		$8>0$ $f_{xx}<0$ MAX.	$24-(-8)^2$ $24-64<0$ SADDLE	$-6-4<0$ SADDLE

$\boxed{\text{(0,-2) is a local max; (-4,6) and (1,-3/2) are saddles.}}$

⑤



dzdydx:

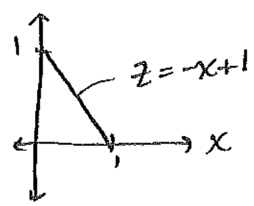


(smash z)

$$\int_0^1 \int_0^{-x+1} \int_0^{1-x-y} (x^2+y^2+z^2)\,dz\,dy\,dx + \int_0^1 \int_{x-1}^{0} \int_0^{1-x+y} (x^2+y^2+z^2)\,dz\,dy\,dx$$

dydzdx:

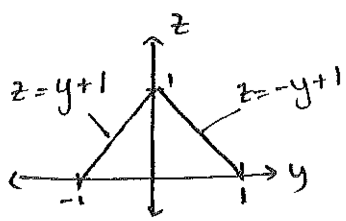
(smash y)



$$\int_0^1 \int_0^{-x+1} \int_{x+z-1}^{1-x-z} (x^2+y^2+z^2)\,dy\,dz\,dx$$

dxdzdy:

(smash x)



$$\int_{-1}^{0} \int_0^{y+1} \int_0^{1+y-z} (x^2+y^2+z^2)\,dx\,dz\,dy$$

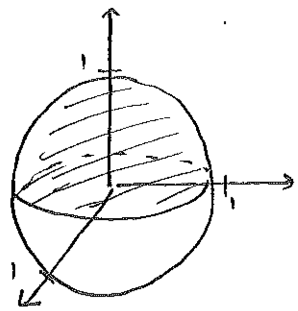
$$+ \int_0^1 \int_0^{-y+1} \int_0^{1-y-z} (x^2+y^2+z^2)\,dx\,dz\,dy$$

(6) (a) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dy\, dx.$

First, draw the region :

$z = 0 \to xy$ - plane

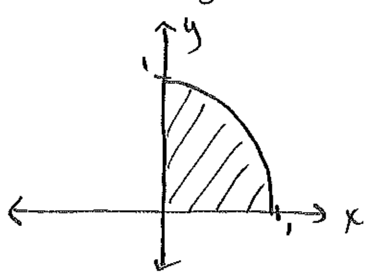
$z = \sqrt{1-x^2-y^2} \to x^2+y^2+z^2=1 \to$ sphere of radius 1



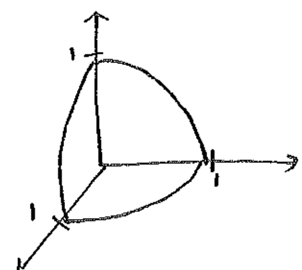
so only the top half of sphere.

After smashing z, we should have :

$0 \le y \le \sqrt{1-x^2}$ and $0 \le x \le 1$ (from the outer 2 $\int$'s)



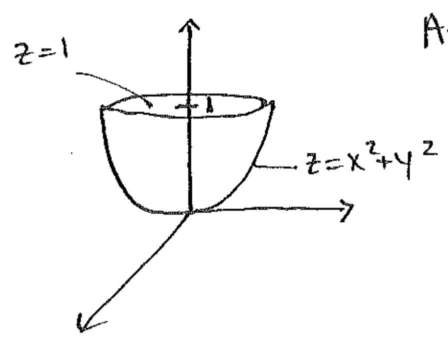
so, we only want one piece of the sphere :



Use spherical, because it's a sphere :

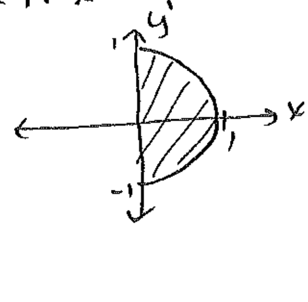
$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$ (see next page for evaluating)

(b) $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{1} dz\, dy\, dx$



After smashing z, we should have :

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ & $0 \le x \le 1$



Use cylindrical, because we have a paraboloid:

$$\int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^1 r\,dz\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 zr \Big|_{z=r^2}^{z=1} dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 (r - r^3)\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right)\Big|_0^1 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \tfrac{1}{2} - \tfrac{1}{4}\,d\theta = \int_{-\pi/2}^{\pi/2} \tfrac{1}{4}\,d\theta = \tfrac{1}{4}\theta \Big|_{-\pi/2}^{\pi/2} = \tfrac{1}{4}\cdot\tfrac{\pi}{2} + \tfrac{1}{4}\cdot\tfrac{\pi}{2}$$

$$= \boxed{\frac{\pi}{4}}$$

Evaluating (a):

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \frac{\rho^5}{5} \sin\phi \Big|_{\rho=0}^{\rho=1} d\phi\, d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \tfrac{1}{5} \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{\pi/2} -\tfrac{1}{5}\cos\phi \Big|_0^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \tfrac{1}{5}\,d\theta$$

$$= \tfrac{1}{5}\theta \Big|_0^{\pi/2}$$

$$= \boxed{\frac{\pi}{10}}$$