Practice Final Solutions

(1) (a) $\vec{F} = \langle \underbrace{2y+2x}_{P}, \underbrace{2x-2y}_{Q} \rangle$, $\vec{x}(t) = \langle t, t^2 \rangle$, $0 \le t \le 1$

$P_y = 2$ Since $\vec{F}$ is defined everywhere & $P_y = 2 = Q_x$,
$Q_x = 2$ $\vec{F}$ is conservative.

$\int 2y+2x \, dx = x^2 + 2xy + C(y)$
$\int 2x-2y \, dy = 2yx - y^2 + D(x)$

$f(x,y) = x^2 + 2xy - y^2$
By the FTC of line integrals, work $= \int_C \vec{F} \cdot d\vec{x} = f(B) - f(A)$
A: $t=0 \rightarrow \vec{x}(0) = \langle 0,0 \rangle \rightarrow A = (0,0)$
B: $t=1 \rightarrow \vec{x}(1) = \langle 1,1 \rangle \rightarrow B = (1,1)$

So work $= f(1,1) - f(0,0) = (1+2-1) - (0+0-0) = \boxed{2}$

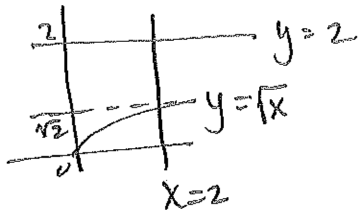
(b) $\vec{F} = \langle \underbrace{x-y}_{P}, \underbrace{x+y}_{Q} \rangle$, $\vec{x}(t) = \langle t, 2t \rangle$, $0 \le t \le 1$

$P_y = 1$, $Q_x = 1$. Since $P_y \ne Q_x$, $\vec{F}$ is not conservative.
$\int_C \vec{F} \cdot d\vec{x} = \int_0^1 \langle t-2t, t+2t \rangle \cdot \langle 1,2 \rangle dt = \int_0^1 \langle -t, 3t \rangle \cdot \langle 1,2 \rangle dt$
$= \int_0^1 -t+6t \, dt = \int_0^1 5t \, dt = \frac{5}{2}t^2 \Big|_0^1 = \boxed{5/2}$

(c) $\vec{F} = \langle 2x, 2y \rangle$, $\vec{x}(t) = \langle \sin(\pi t), e^{t^2-t} \rangle$, $0 \le t \le 1$
Since $\vec{F} = \vec{\nabla}(x^2+y^2)$, $\vec{F}$ is conservative.

$\vec{x}(0) = \langle 0, 1 \rangle$
$\vec{x}(1) = \langle 0, 1 \rangle$
Since the initial & terminal pts are = (i.e. $\vec{x}(t)$ is a closed curve), by the FTC for line integrals, work $= \boxed{0}$.

② (a)



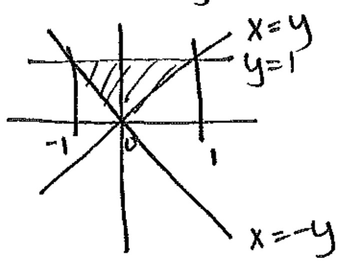
$$\int_0^2\int_{\sqrt{x}}^{2} y\,dy\,dx = \int_0^{\sqrt{2}}\int_0^{y^2} y\,dx\,dy + \int_{\sqrt{2}}^{2}\int_0^{2} y\,dx\,dy$$

$$= \int_0^{\sqrt{2}} xy\Big|_0^{y^2} dy + \int_{\sqrt{2}}^{2} xy\Big|_0^{2} dy$$

$$= \int_0^{\sqrt{2}} y^3\,dy + \int_{\sqrt{2}}^{2} 2y\,dy = \frac{y^4}{4}\Big|_0^{\sqrt{2}} + y^2\Big|_{\sqrt{2}}^{2}$$

$$= 1 + (4-2) = 1+2 = \boxed{3}$$

(b) $$\int_0^1\int_{-y}^{y} x^2\,dx\,dy = \int_{-1}^{0}\int_{-x}^{1} x^2\,dy\,dx + \int_0^1\int_x^1 x^2\,dy\,dx$$



$$= \int_{-1}^{0} x^2 y\Big|_{-x}^{1} dx + \int_0^1 x^2 y\Big|_x^1 dx$$

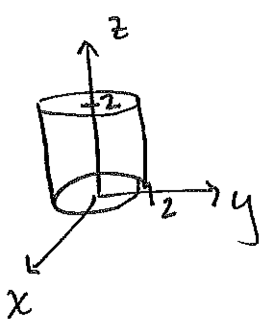
$$= \int_{-1}^{0} x^2 + x^3\,dx + \int_0^1 x^2 - x^3\,dx$$

$$= \left(\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_{-1}^{0} + \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1$$

$$= \left(0 - \left(\frac{-1}{3} + \frac{1}{4}\right)\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{3} - \frac{1}{4} = \frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$$

③



(with $0 \le z \le 2$. See next pg for $0 \le z \le 3$)

$$\iiint_D (x^2+y^2-z^2)\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^2\int_0^2 (r^2-z^2)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_0^2 r^3 - rz^2\,dz\,dr\,d\theta$$

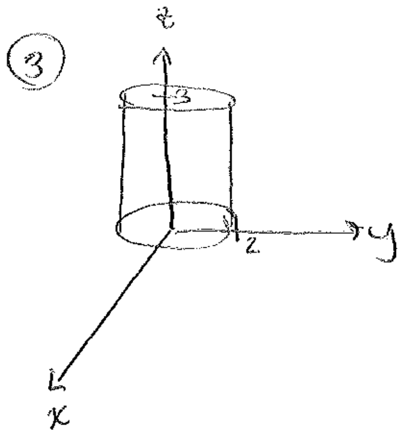
$$= \int_0^{2\pi}\int_0^2 r^3 z - \frac{rz^3}{3}\Big|_0^2 dr\,d\theta = \int_0^{2\pi}\int_0^2 2r^3 - \frac{8}{3}r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{4} - \frac{8}{6}r^2\Big|_0^2 d\theta = \int_0^{2\pi} 4 - 16/3\,d\theta$$

$$= -\frac{4}{3}(2\pi) = \boxed{-\frac{8}{3}\pi}$$

(with $0 \leq z \leq 3$)

(3)



$$\iiint_D (x^2+y^2-z^2)\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^2\int_0^3 (r^2-z^2)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_0^3 r^3 - z^2 r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3 z - \frac{rz^3}{3}\Big|_0^3\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 3r^3 - 9r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{3r^4}{4} - \frac{9r^2}{2}\Big|_0^2\,d\theta$$

$$= \int_0^{2\pi} 12 - 18\,d\theta = -6\int_0^{2\pi} d\theta$$

$$= \boxed{-12\pi}$$

④ $f(x,y) = e^{x-2y}$

(a) $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$

$\left.\frac{\partial f}{\partial x}\right|_{(2,1)} = e^{x-2y}\Big|_{(2,1)} = e^{2-2} = 1$

$\left.\frac{\partial f}{\partial y}\right|_{(2,1)} = -2e^{x-2y}\Big|_{(2,1)} = -2e^{2-2} = -2$

$\frac{dy}{dx} = -\left(\frac{1}{-2}\right) = \boxed{\frac{1}{2}}$

(b) $z=1$ at $(2,1)$

$\boxed{y-1 = 1/2\,(x-2)}$

OR

$\vec{r}(t) = \langle 2,1\rangle + t\langle 1,-2\rangle = \langle 2+t, 1-2t\rangle$.

⑤ $f(x,y) = 2xy - x^2 - y$, $0 \le x \le 1$, $0 \le y \le 1$.

Interior:

$\vec{\nabla} f(x,y) = \langle 2y-2x, 2x-1\rangle = \langle 0,0\rangle$

$2x = 1 \rightarrow x = 1/2$

$2y - 2(1/2) = 0 \rightarrow 2y - 1 = 0 \rightarrow y = 1/2$

Boundary:

$x=0$: $f_1(y) = -y$

$f_1'(y) = -1 \neq 0$

$x=1$: $f_2(y) = 2y - y - 1 = y - 1$

$f_2'(y) = 1 \neq 0$

$y=0$: $f_3(x) = -x^2$

$f_3'(x) = -2x = 0$

$x = 0$

$y=1$: $f_4(x) = 2x - x^2 - 1$

$f_4'(x) = 2 - 2x = 0$

$x = 1$

(x,y)	$f(x,y)$
$(1/2, 1/2)$	$-1/4$
$(0,0)$	0
$(1,0)$	-1
$(0,1)$	-1
$(1,1)$	0

Max: $(0,0)$ & $(1,1)$
Min: $(1,0)$ & $(0,1)$