Worksheet 3 Solutions

(3) $x(t) = \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \end{pmatrix}$

(a) $v(t) = x'(t) = \begin{pmatrix} -R\omega\sin(\omega t) \\ R\omega\cos(\omega t) \end{pmatrix}$

$$\begin{aligned} \|v(t)\| &= \sqrt{R^2\omega^2\sin^2(\omega t) + R^2\omega^2\cos^2(\omega t)} \\ &= \sqrt{R^2\omega^2(\sin^2(\omega t) + \cos^2(\omega t))} \\ &= \sqrt{R^2\omega^2} \\ &= R\omega \end{aligned}$$

[notice that the problem states that $R, \omega \geq 0$, or else this would be $|R||\omega|$.]

(b) $a(t) = x''(t) = \begin{pmatrix} -R\omega^2\cos(\omega t) \\ -R\omega^2\sin(\omega t) \end{pmatrix} = -\omega^2\begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \end{pmatrix} = -\omega^2 x(t).$

(c) $F = ma$

$= m(-\omega^2 x(t))$

$\|F\| = \|m(-\omega^2 x(t))\|$

$\|F\| = m\omega^2\|x(t)\|$ $\qquad \|x(t)\| = \sqrt{R^2\cos^2(\omega t) + R^2\sin^2(\omega t)} = R$

$= m\omega^2 R$

$= \dfrac{m\omega^2R^2}{R} = \boxed{\dfrac{m\|v\|^2}{R}}$

[Note: $m\omega\|v\|$ would be ok, too, but this is the usual formula]

(4) $x(t) = \begin{pmatrix} e^t\cos t \\ e^t\sin t \end{pmatrix}$

(a) $(1,0): \begin{cases} e^t\cos t = 1 \\ e^t\sin t = 0 \end{cases} \Rightarrow t = 0$

$(-e^{\pi}, 0): \begin{cases} e^t\cos t = -e^{\pi} \\ e^t\sin t = 0 \end{cases} \Rightarrow t = \pi$

(b) $\int_0^{\pi} \| x'(t) \| \, dt.$

$$x'(t) = \begin{pmatrix} e^t \cos t - e^t \sin t \\ e^t \sin t + e^t \cos t \end{pmatrix} = e^t \begin{pmatrix} \cos t - \sin t \\ \sin t + \cos t \end{pmatrix}$$

$$\|x'(t)\| = \sqrt{e^{2t}\left((\cos t - \sin t)^2 + (\sin t + \cos t)^2\right)}$$

$$= e^t \sqrt{\cos^2 t - \cancel{2\cos t \sin t} + \sin^2 t + \sin^2 t + \cancel{2\cos t \sin t} + \cos^2 t}$$

$$= e^t \sqrt{2}$$

$$\int_0^{\pi} \sqrt{2}\, e^t \, dt = \sqrt{2} \int_0^{\pi} e^t \, dt = \sqrt{2}\, e^t \Big|_0^{\pi} = \boxed{\sqrt{2}\, e^{\pi} - \sqrt{2}}$$

(c) we want to find the value of $t = t_0$ so that

$$\int_0^{t_0} \|x'(t)\| \, dt = \frac{\sqrt{2}\, e^{\pi} - \sqrt{2}}{2}.$$

$$\int_0^{t_0} \sqrt{2}\, e^t \, dt = \frac{\sqrt{2}\, e^{\pi} - \sqrt{2}}{2}$$

$$\sqrt{2}\, e^t \Big|_0^{t_0} = \frac{\sqrt{2}\, e^{\pi} - \sqrt{2}}{2}$$

$$\sqrt{2}\, e^{t_0} - \sqrt{2} = \frac{\sqrt{2}}{2} e^{\pi} - \frac{\sqrt{2}}{2}$$

$$\sqrt{2}\, e^{t_0} = \frac{\sqrt{2}}{2} e^{\pi} + \frac{\sqrt{2}}{2}$$

$$e^{t_0} = \frac{1}{2} e^{\pi} + \frac{1}{2}$$

$$t_0 = \ln\left(\tfrac{1}{2} e^{\pi} + \tfrac{1}{2}\right).$$

The pt on the curve:

$$x\left(\ln\left(\tfrac{1}{2}e^{\pi} + \tfrac{1}{2}\right)\right) = \begin{pmatrix} e^{\ln(\frac{1}{2}e^{\pi} + \frac{1}{2})} \cos\left(\ln\left(\tfrac{1}{2}e^{\pi} + \tfrac{1}{2}\right)\right) \\ e^{\ln(\frac{1}{2}e^{\pi} + \frac{1}{2})} \sin\left(\ln\left(\tfrac{1}{2}e^{\pi} + \tfrac{1}{2}\right)\right) \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \left(\tfrac{1}{2}e^{\pi} + \tfrac{1}{2}\right) \cos\left(\ln\left(\tfrac{1}{2}e^{\pi} + \tfrac{1}{2}\right)\right) \\ \left(\tfrac{1}{2}e^{\pi} + \tfrac{1}{2}\right) \sin\left(\ln\left(\tfrac{1}{2}e^{\pi} + \tfrac{1}{2}\right)\right) \end{pmatrix}}$$