Solutions to Series Convergence Worksheet on Homework #10

Note: There is often more than one way to determine the convergence of a series. These are not the only solutions to these problems.

1. $\sum_{n=1}^{\infty} \frac{11}{n^5}$.

   We will use the Integral Test, with $f(x) = \frac{11}{x^5}$.
   Hypotheses:

   - $\frac{11}{x^5}$ is positive and decreasing, because $1 \le x < \infty$.
   - $f(k) = \frac{11}{k^5}$ for all $k = 1, 2, \ldots$

   Thus, by the integral test, $\sum_{n=1}^{\infty} \frac{11}{n^5}$ converges if and only if the integral $\int_1^{\infty} \frac{11}{x^5}\,dx < \infty$. To determine the convergence of $\int_1^{\infty} \frac{11}{x^5}\,dx$, we can reference the chart. $\int_1^{\infty} \frac{11}{x^5}\,dx = 11 \int_1^{\infty} \frac{1}{x^5}\,dx < \infty$ because $5 > 1$. Therefore, our series $\sum_{n=1}^{\infty} \frac{11}{n^5}$ also converges.

2. $\sum_{n=1}^{\infty} \frac{1}{2n^{1/2}+1}$

   We will use the Limit Comparison Test, with $b_n = \frac{1}{n^{1/2}}$ (Note that you could also use $b_n = \frac{1}{2n^{1/2}}$).
   Hypotheses:

   - $\frac{1}{2n^{1/2}+1}$ is positive (because all pieces are positive) and decreasing (because the numerator is constant while the denominator increases), and the same is true for $\frac{1}{n^{1/2}}$ (for the same reasons).
   - $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2n^{1/2}+1} \cdot \frac{n^{1/2}}{1} = \frac{1}{2},$$
     which is neither 0 nor ∞.

   Thus, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2n^{1/2}+1}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ does.
   To determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, we will use the Integral Test with $f(x) = \frac{1}{x^{1/2}}$.
   Hypotheses:

   - $f(x)$ is positive and decreasing, as in reasoning above, for the Limit Comparison Test.
   - $f(k) = a_k$ for all $k = 1, 2, \ldots$

   Thus, by the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ converges if and only if $\int_1^{\infty} \frac{1}{x^{1/2}}\,dx < \infty$. By the chart, $\int_1^{\infty} \frac{1}{x^{1/2}}\,dx$ diverges, because $\frac{1}{2} \not> 1$. Therefore, by the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges.
   Finally, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{2n^{1/2}+1}$ diverges, as well.

3. $\sum_{n=}^{\infty} \frac{n!}{(2n)!}$

   I will use the Ratio Test.

   $$\lim_{n\to\infty} \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} = \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = 0$$

Thus, $L = 0 < 1$, and so by the Ratio Test, we conclude that the series converges.

4. $\sum_{n=1}^{\infty} \frac{3^{n+4}}{n!}$
We will use the Ratio Test.

$$\lim_{n\to\infty} \frac{3^{n+5}}{(n+1)!} \cdot \frac{n!}{3^{n+4}} = \lim_{n\to\infty} \frac{3}{n+1} = 0$$

Thus, $L = 0 < 1$, and so by the Ratio Test, we conclude that the series converges.

5. $\sum_{n=1}^{\infty} \left(\frac{n+1}{n^2+5}\right)^n$
We will use the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n+1}{n^2+5}\right|^n} = \lim_{n\to\infty} \left|\frac{n+1}{n^2+5}\right| = \lim_{n\to\infty} \frac{n+1}{n^2+5} = 0$$

Thus, $L = 0 < 1$, and so by the Root Test, we conclude that the series converges.

6. $\sum_{n=1} \infty \frac{1}{n2^n}$
We will use the Ratio Test.

$$\lim_{n\to\infty} \frac{1}{(n+1)2^{n+1}} \cdot \frac{n2^n}{1} = \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2}$$

Thus, $L = \frac{1}{2} < 1$, and so by the Ratio Test, we conclude that the series converges.

7. $\sum_{n=1}^{\infty} \frac{3+2^{-n}}{\sqrt{n}}$
We will use the Limit Comparison Test, with $b_n = \frac{3}{\sqrt{n}}$. (Notice that 3 is the term in the numerator that increases the fastest, since 2^{-n} actually decreases.)
Hypotheses:

   - a_n is positive (the numerator is positive because it is the sum of positive numbers, and the denominator is positive because square roots are always positive) and decreasing (because in the numerator, 3 is constant and 2^{-n} is decreasing, while in the denominator, $\sqrt{n}$ is increasing). b_n is also positive and decreasing, for similar reasons.
   - $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3+2^{-n}}{\sqrt{n}} \cdot \frac{\sqrt{n}}{3} = \lim_{n\to\infty} \frac{3+2^n}{3} = 1 + \lim_{n\to\infty} \frac{2^{-n}}{3} = 1 + 0 = 1,$$
     and this limit is not 0 or ∞.

Therefore, by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{3+2^{-n}}{\sqrt{n}}$ converges if and only $sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ converges.

To determine the convergence of $sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$, we will use the integral test, with $f(x) = \frac{3}{\sqrt{x}}$.
Hypotheses:

- $f(x)$ is positive and decreasing.
- $f(k) = \frac{3}{\sqrt{k}}$ for all $k = 1, 2, \ldots$

Thus, by the Integral Test, $sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ converges if and only if $\int_1^\infty \frac{3}{\sqrt{x}}\,dx < \infty$. From the chart, we know that $\int_1^\infty \frac{3}{\sqrt{x}}\,dx$ is not finite, because $\frac{1}{2} \not> 1$. Therefore, $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ diverges.

Finally, from the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{3+2^{-n}}{\sqrt{n}}$ diverges, as well.

8. $\sum_{n=1}^{\infty} \frac{3n^{5/2}}{\sqrt{n^5+n^2+1}}$

   We will use the Term Test.

$$\lim_{n\to\infty} \frac{3n^{5/2}}{\sqrt{n^5+n^2+1}} \cdot \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} = \lim_{n\to\infty} \frac{3}{\sqrt{1+\frac{n^2}{n^5}+\frac{1}{n^5}}} = 3$$

   Since $3 \neq 0$, by the Term Test, the series diverges.