Review Sheet Solutions

① $\sqrt{(3)^2+(4)^2+(-2)^2} = \sqrt{9+16+4} = \sqrt{29}$

$$\boxed{\begin{pmatrix} 3/\sqrt{29} \\ 4/\sqrt{29} \\ -2/\sqrt{29} \end{pmatrix}}$$

② If u & v are parallel, then $u = kv$, for some scalar $k \in \mathbb{R}$.

$$\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = k\begin{pmatrix} -8 \\ 4 \\ -16 \end{pmatrix} \longrightarrow \begin{cases} 2 = -8k \longrightarrow k = -1/4 \\ -1 = 4k \longrightarrow k = -1/4 \\ 4 = -16k \longrightarrow k = -1/4 \end{cases} \checkmark$$

yes, u & v are parallel. (so they are not perpendicular)

If u & v are perpendicular, then $u \cdot v = 0$.

$$\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 4 \\ -16 \end{pmatrix} = -16 - 4 - 64 \neq 0.$$

③ $A(0,2,4)$ $B(3,-1,6)$ $C(2,5,4)$

(a) $\vec{AB} = \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$

$\vec{AC} = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 3 & -3 & 2 \\ 2 & 3 & 0 \end{vmatrix}$$

$= i(-6) - j(4) + k(9+6)$

$= -6i - 4j + 15k$, so

the normal vector is $n = \begin{pmatrix} -6 \\ -4 \\ 15 \end{pmatrix}$

A plane has the equation $\vec{n} \cdot \vec{AX} = 0$, so

$$\begin{pmatrix} -6 \\ -4 \\ 15 \end{pmatrix} \cdot \begin{pmatrix} x \\ y-2 \\ z-4 \end{pmatrix} = 0$$

$$\boxed{-6x - 4(y-2) + 15(z-4) = 0}$$

(b) area (ΔABC) = $\frac{1}{2}$ area(parallelogram spanned by $\vec{AB}$ & $\vec{AC}$)

$= \frac{1}{2} \cdot \| \vec{AB} \times \vec{AC} \|$

$= \frac{1}{2} \cdot \left\| \begin{pmatrix} -6 \\ -4 \\ 15 \end{pmatrix} \right\|$

$= \frac{1}{2}\sqrt{36 + 16 + 225}$

$= \boxed{\frac{1}{2}\sqrt{277}}$

(c) direction of line is $\vec{AB}$.

$$\boxed{l(t) = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}}$$

④ $u \cdot v = \|u\| \cdot \|v\| \cdot \cos\theta$

$\begin{pmatrix}3\\2\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\0\\4\end{pmatrix} = \left(\sqrt{9+4+1}\right)\left(\sqrt{1+0+16}\right)\cos\theta$

$-3+0+4 = \left(\sqrt{14}\right)\left(\sqrt{17}\right)\cos\theta$

$\cos\theta = \dfrac{1}{\sqrt{14 \cdot 17}}$

$\boxed{\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{14 \cdot 17}}\right)}$

⑤ $x(t) = \begin{pmatrix}2\cos(-2t)\\2\sin(-2t)\end{pmatrix}$

⑥ $v(t) = x'(t) = \boxed{\begin{pmatrix}4\sin(-2t)\\-4\cos(-2t)\end{pmatrix} = \text{velocity}}$

Speed $= \|v(t)\| = \sqrt{16\sin^2(-2t)+16\cos^2(-2t)} = \sqrt{16\left(\cos^2(-2t)+\sin^2(-2t)\right)} = \sqrt{16}$
$= \boxed{4}$

$\boxed{T(t) = \dfrac{v(t)}{\|v(t)\|} = \begin{pmatrix}\sin(-2t)\\-\cos(-2t)\end{pmatrix}}$

$N(t) = \dfrac{T'(t)}{\|T'(t)\|}$. $T'(t) = \begin{pmatrix}-2\cos(-2t)\\2\sin(-2t)\end{pmatrix}$

$\|T'(t)\| = \sqrt{4\cos^2(-2t)+4\sin^2(-2t)} = 2.$

$\boxed{N(t) = \begin{pmatrix}-\cos(-2t)\\\sin(-2t)\end{pmatrix}}$

⑦ They are perpendicular. (we proved this on worksheet 3)

⑧ $\kappa = \dfrac{1}{\|x'(t)\|} \cdot \|T'(t)\|$

$x'(t) = \begin{pmatrix}-\sin t\\\cos t\\2\end{pmatrix} \longrightarrow \|x'(t)\| = \sqrt{\sin^2 t+\cos^2 t+4} = \sqrt{5}$

$T(t) = \dfrac{1}{\sqrt{5}}\begin{pmatrix}-\sin t\\\cos t\\2\end{pmatrix} \longrightarrow T'(t) = \dfrac{1}{\sqrt{5}}\begin{pmatrix}-\cos t\\-\sin t\\0\end{pmatrix}$

$\|T'(t)\| = \dfrac{1}{\sqrt{5}} \cdot \sqrt{\cos^2 t+\sin^2 t+0} = \dfrac{1}{\sqrt{5}} \cdot 1 = \dfrac{1}{\sqrt{5}}$

$\boxed{\kappa = \left(\dfrac{1}{\sqrt{5}}\right) \cdot \left(\dfrac{1}{\sqrt{5}}\right) = \dfrac{1}{5}}$

This helix is twice as loosely coiled. That is, this helix wraps around once in the same time as it takes the standard helix to wrap around twice.

(9) $\begin{pmatrix} \cos t \\ \sin t \\ 2t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4\pi \end{pmatrix} \rightsquigarrow t = 2\pi$

$$x'(2\pi) = \begin{pmatrix} -\sin(2\pi) \\ \cos(2\pi) \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\boxed{l(s) = \begin{pmatrix} 1 \\ 0 \\ 4\pi \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}$$

This line intersects the xy-plane when the z-component is zero:

$$4\pi + 2s = 0$$
$$2s = -4\pi$$
$$s = -2\pi$$

So the point is:

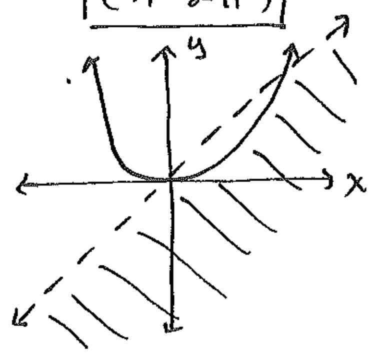
$$l(-2\pi) = \begin{pmatrix} 1 \\ 0 \\ 4\pi \end{pmatrix} - 2\pi\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2\pi \\ 0 \end{pmatrix}$$

$$\boxed{(1, -2\pi)}$$

(10) $f(x,y) = \sqrt{x^2 - y} + \ln(x-y)$.

$x^2 - y \geq 0$ AND $x - y > 0$

$y \leq x^2$ AND $y < x$



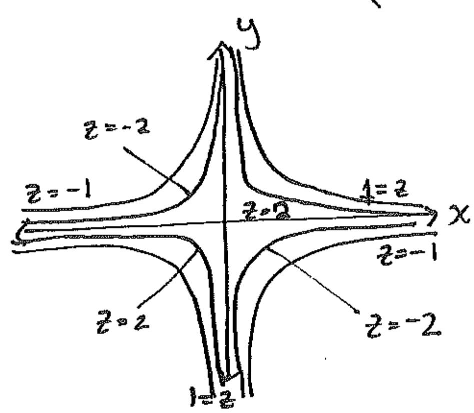
(11) $f(xy) = \frac{1}{xy}$

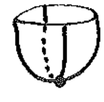
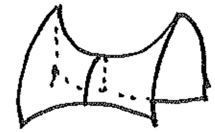
$z=1$: $1 = \frac{1}{xy}$, $y = \frac{1}{x}$

$z=2$: $2 = \frac{1}{xy}$, $y = \frac{1}{2x}$

$z=-1$: $-1 = \frac{1}{xy}$, $y = -1/x$

$z=-2$: $-2 = \frac{1}{xy}$, $y = -\frac{1}{2x}$



(12) $f(x,y) = x^2+y^2$  $g(x,y) = x^2-y^2$ 

$h(x,y) = \sqrt{x^2+y^2}$  $k(x,y) = x^2$ 

(13) f has the same shape as g (paraboloid), but it is shifted up 4 units.

(14) $f(x,y) = x^2y - \sin(x^2+y)$

$$\boxed{\begin{aligned} f_x &= 2xy - 2x\cos(x^2+y) \\ f_y &= x^2 - \cos(x^2+y) \end{aligned}}$$

(15) $xyz = 1$, $(2,1)$

$z = \frac{1}{xy} = f(x,y)$

$f_x = \frac{-y}{(xy)^2} \rightarrow f_x(2,1) = \frac{-1}{(2)^2} = -\frac{1}{4}$

$f_y = \frac{-x}{(xy)^2} \rightarrow f_y(2,1) = \frac{-2}{(2)^2} = -\frac{1}{2}$

$f(2,1) = \frac{1}{2 \cdot 1} = \frac{1}{2}$

$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$$\boxed{z = \frac{1}{2} - \frac{1}{4}(x-2) - \frac{1}{2}(y-1)}$$

(16) Using the tangent plane in #16:

$z \approx \frac{1}{2} - \frac{1}{4}(1.9-2) - \frac{1}{2}(1.01-1)$

$\approx \frac{1}{2} - \frac{1}{4}(-.1) - \frac{1}{2}(.01)$

$\approx 0.5 - 0.25(-.1) - 0.5(0.01)$

$\approx 0.5 + 0.025 - 0.005$

$\approx \boxed{0.52}$

(17) $f(x,y) = \frac{1}{xy}$, $\gamma(t) = \begin{pmatrix} t^2 \\ t \end{pmatrix}$

$(f \circ \gamma)(t) = \frac{1}{(t^2)(t)} = \frac{1}{t^3}$

$(f \circ \gamma)'(t) = -3t^{-4}$

$$\boxed{(f \circ \gamma)'(1) = -3}$$

If you move one unit, time. along the path γ, the surface goes down 3 units.