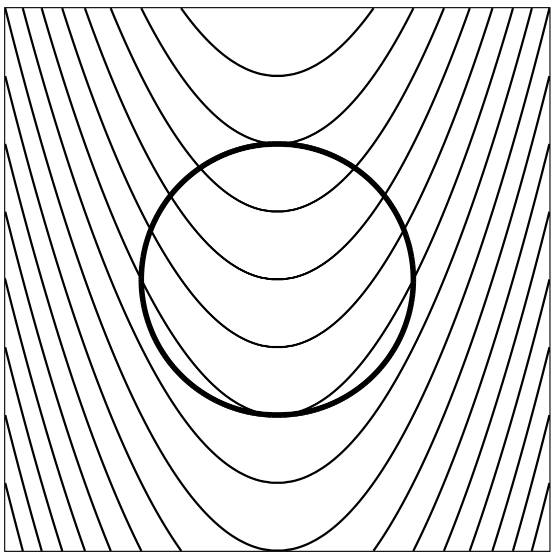
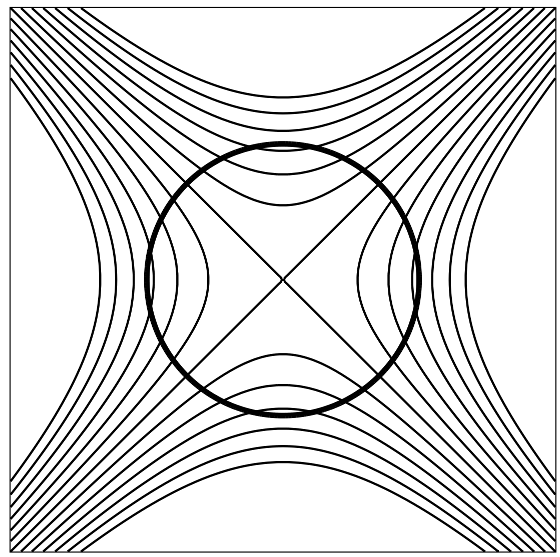
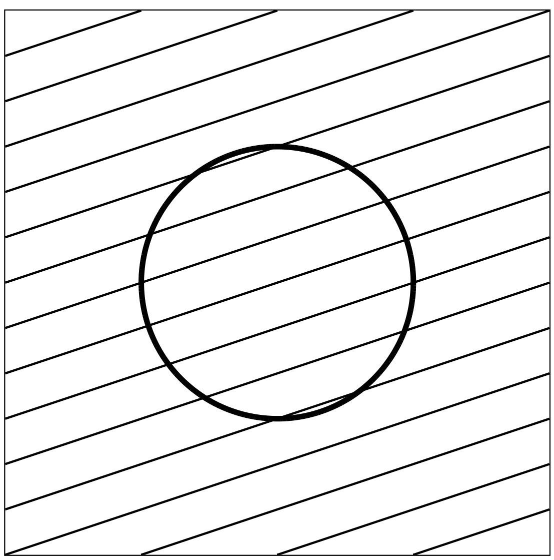
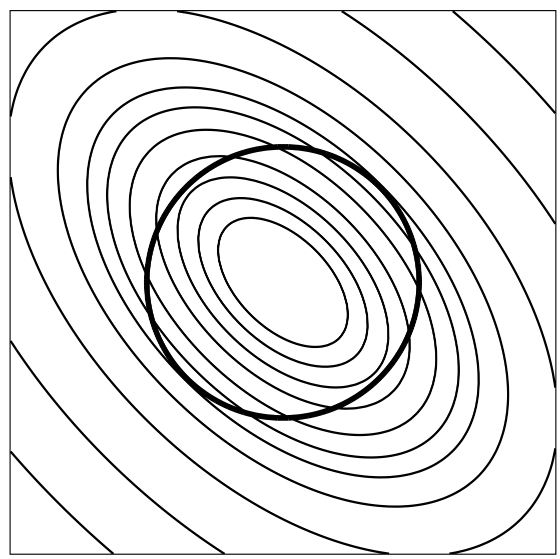
1. **(a)** Consider each of the eight figures below. In each figure, the thick circle is the curve $g(x, y) = c$. The other curves in that figure are level sets for a function of the form $f(x, y)$. For each figure, mark the (approximate) locations of all the solutions of the Lagrange system

$$\begin{cases} \nabla f(x,y) = \lambda \ \nabla g(x,y), \\ g(x,y) = c. \end{cases}$$

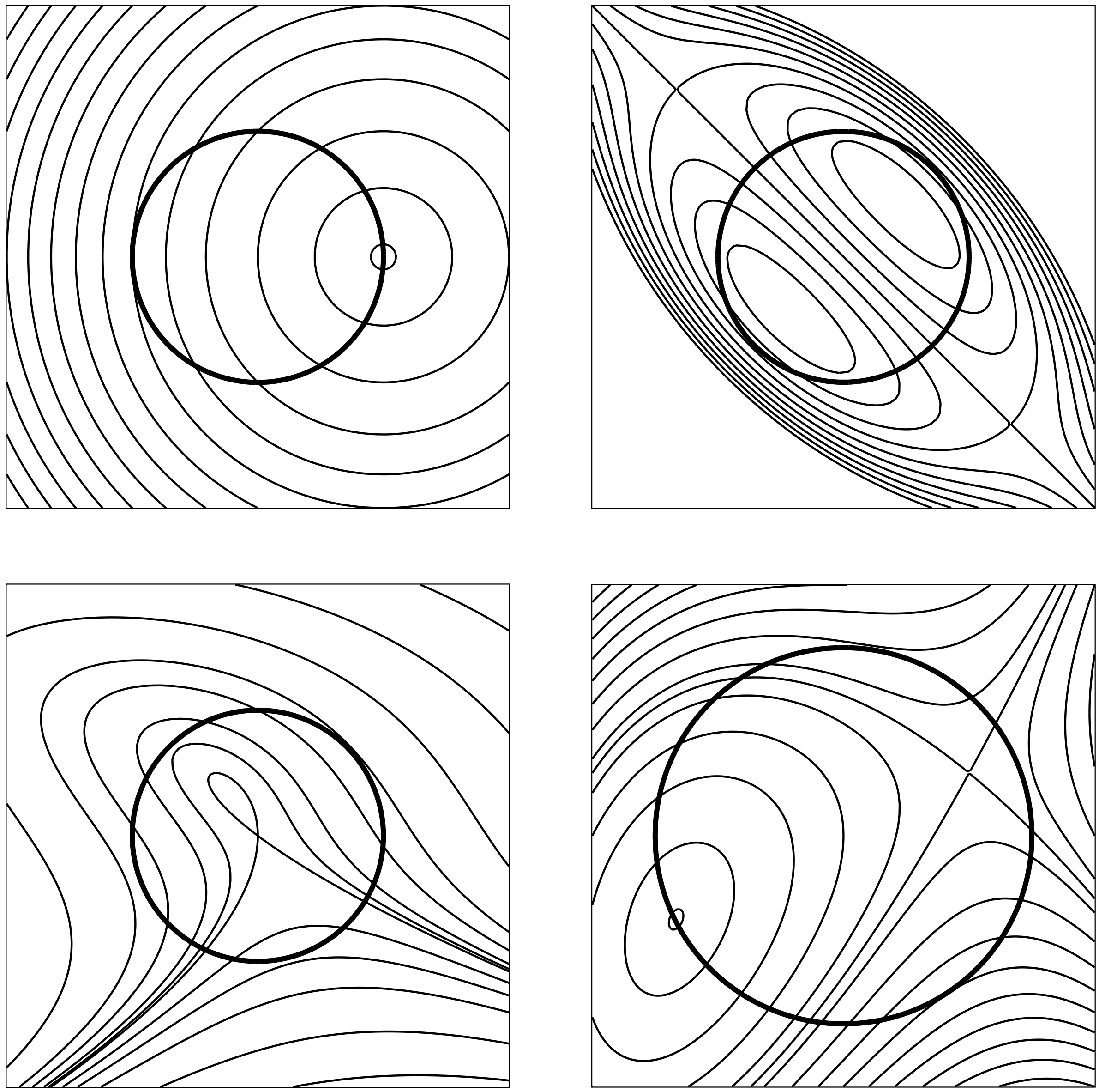








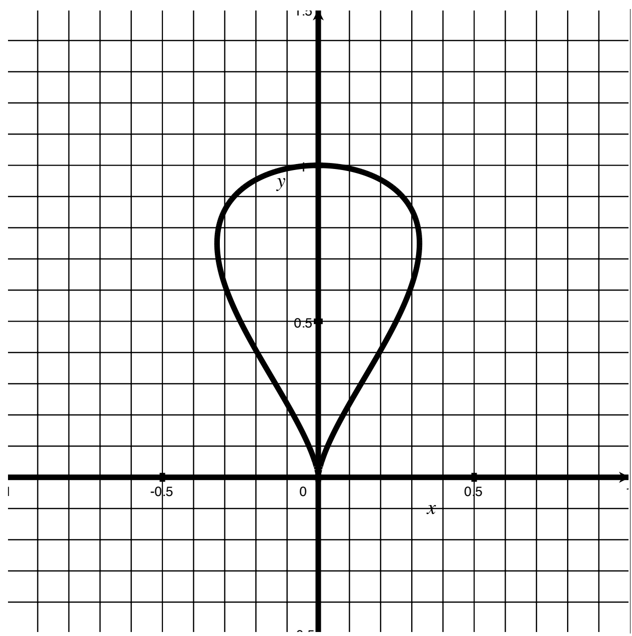
(more figures on the back ...)



(b) Referring to each of the figures above, is there any guarantee that $f(x, y)$ actually attains a global maximum value and a global minimum value somewhere on the constraint $g(x, y) = c$? Why or why not?

(c) For each of the locations that you marked in the figures, can you decide if it is a maximum point, a minimum point, or neither? Explain why or why not. If the answer is no, what additional information would you need, and what would you do with that information?

2. Consider the curve $g(x, y) = x^2 - y^3 + y^4 = 0$. Its graph is the loop in the following figure.



Let $f(x, y) = x - y$.

(a) Without doing any computations, is there any guarantee that $f(x, y)$ actually attains a global minimum value and a global maximum value somewhere on the constraint $g(x, y) = 0$? Explain!

(b) In the above figure, draw an assortment of level sets for $f(x, y)$. (Label the levels.)

(c) Mark the **approximate** location where $f(x, y)$ attains its minimum value on $g(x, y) = 0$.

(d) Mark the **exact** location where $f(x, y)$ attains its maximum value on $g(x, y) = 0$.

(e) Set up (but don't solve!) the Lagrange system $\begin{cases} \nabla f(x, y) = \lambda \, \nabla g(x, y), \\ g(x, y) = 0. \end{cases}$

(f) Is $(0, 0)$ a solution of the Lagrange system from **(e)**?

(g) What is $\nabla g(0, 0)$?

(h) What is the moral of this problem?

3. For each function, find the global maximum and minimum on the given domain.

   (a) $f(x, y, z) = xyz$ on $x + y + z = 1$, if $x \geq 0, y \geq 0, z \geq 0$.

   (b) $f(x, y, z) = x + y + 2z$ on $x^2 + y^2 + z^2 \leq 3$.

   (c) $f(x, y, z) = x^2 - y^2$ on $x^2 + 2y^2 + 3z^2 \leq 1$.

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4. Find the dimensions of the box with the largest volume if the total surface area is 64 cm^2.

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5. (a) Use Lagrange multipliers to show that $f(x, y, z) = z^2$ has only one critical point on the surface $x^2 + y^2 - z = 0$.

   (b) Show that the one critical point is a minimum.

   (c) Sketch the surface. Why did Lagrange multipliers not find a maximum of f on the surface?

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6. A consumer has \$600 to spend on two commodities, the first of which costs \$20 per unit and the second \$30 per unit. Suppose the utility derived by the consumer from x units of the first commodity and y units of the second commodity is given by the $U(x, y) = 10x^{0.6}y^{0.4}$. How many units of each commodity should the consumer buy to maximize utility?