1. For each of the following vector fields, can you find a function f such that ∇f is the given vector field? If so, find such a function. If not, explain why not. Recall that such a function is called a **potential function** for the vector field.

   (a) $\begin{pmatrix} 6xy^2 \\ 6x^2y \end{pmatrix}$

   (b) $\begin{pmatrix} x \\ y \end{pmatrix}$

   (c) $\begin{pmatrix} y \\ x \end{pmatrix}$

   (d) $\begin{pmatrix} -y \\ x \end{pmatrix}$

   (e) $\begin{pmatrix} y\sin x + xy\cos x \\ x\sin x \end{pmatrix}$

   (f) $\begin{pmatrix} 2xy^3 + \sin y \\ 3x^2y^2 + \cos y \end{pmatrix}$

   (g) $\begin{pmatrix} ye^{xy} + xy^2e^{xy} \\ xe^{xy} + xye^{xy} \end{pmatrix}$

   (h) $\begin{pmatrix} y^2\cos(xy^2) \\ 2xy\cos(xy^2) \end{pmatrix}$

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2. Evaluate the following **line integrals of vector fields** using the definition

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_{t=a}^{t=b} \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\ dt$$

   where $\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ is a parametrization of a curve with $a \leq t \leq b$.

   (a) $\int_\gamma \begin{pmatrix} 3y \\ 2x \\ 4z \end{pmatrix} \cdot d\mathbf{s}$ where γ parametrizes the straight-line segment from $(0,0,0)$ to $(1,1,1)$.

   (b) Integrate $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$ along the curve $x = y^2$ from $(0,0)$ to $(1,1)$.

   (c) Integrate $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$ along the straight-line segment from $(0,0)$ to $(1,1)$.

   (d) Integrate $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ counter-clockwise along the unit circle $x^2 + y^2 = 1$ from $(1,0)$ to $(0,1)$.

   (e) Integrate $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ along the straight-line segment from $(1,0)$ to $(0,1)$.

3. Find the work done by the force $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$ as a particle moves from $(1, 1)$ to $(2, 3)$ along a straight line. Recall that the work W done by F over a curve C is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

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4. Sometimes line integrals of vector fields are presented to you in different ways, though we always use the same method of evaluation. When we write

$$\int_C P\ dx + Q\ dy$$

for some functions P and Q, then the vector field we're integrating is $\mathbf{F} = \langle P, Q \rangle$. If we think of $d\mathbf{r} = \langle dx, dy \rangle$, then we see that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \langle P, Q \rangle \cdot \langle dx, dy \rangle = \int_C P\ dx + Q\ dy.$$

(This is a little mathematically dubious, but gives a good justification for how the symbols fit together)

Using this, evaluate $\int_C xy\ dx + (x + y)\ dy$ where C is the curve $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

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5. Evaluate the following **line integrals of scalar functions** using the definition

$$\int_C f(x, y, z)\ ds = \int_{t=a}^{t=b} f(\mathbf{r}(t))\ \|\mathbf{r}'(t)\|\ \ dt$$

where $\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ is a parametrization of the curve C with $a \leq t \leq b$.

(a) $\int_C (x + y)\ ds$ where C is the straight-line segment from $(0, 1, 0)$ to $(1, 0, 0)$.

(b) $\int_C \sqrt{x^2 + y^2}\ ds$ where C is the helix parametrized by $\mathbf{r}(t) = \begin{pmatrix} 4\cos t \\ 4\sin t \\ 3t \end{pmatrix}$ for $-2\pi \leq t \leq 2\pi$.

(c) (The answer to this problem is not zero!) Integrate $f(x, y, z) = -\sqrt{x^2 + z^2}$ over the circle $\mathbf{r}(t) = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k}$ with $0 \leq t \leq 2\pi$

(d) $\int_C \sqrt{x + 2y}\, ds$ over the straight-line segment from $(1, 0)$ to $(1, 2)$.

(e) $\int_C \frac{1}{x^2+y^2+1}\ ds$ where C is the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

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6. A vector field $\mathbf{F}$ is called **conservative** if there is some function f such that $\nabla f = \mathbf{F}$. Such a function f is called a **potential function**. Line integrals of conservative vector fields can be computed by the following elegant formula.

   Suppose C is any (smooth) path from a point A to a point B. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

   In particular, observe that the value of the integral doesn't depend on how we get from A to B. This means we can skip finding a parametrization for C, which is often the most difficult part.

   (a) Integrate $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$ where C is any (smooth) path from $(0,0)$ to $(1,1)$. What if C is from $(0,0)$ to (a,b)?

   (b) Using your work from problem 2 on Worksheet 19, is $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ a conservative vector field?

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7. Consider the field

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 1+y \\ x \end{pmatrix}.$$

   Compute the work done by $\mathbf{F}$ along each of the following curves:

   (a) The straight line segment from $(-1,-1)$ to $(1,1)$.

   (b) The part of the curve $y = x^3$ from $(-1,-1)$ to $(1,1)$.

   (c) The unit circle, oriented counterclockwise. (*Hint: use* $\cos^2(t) - \sin^2(t) = \cos(2t)$.)

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8. In the previous problem, is there anything interesting about the answers you got? Was this just a coincidence, or is there a deeper reason why this happened?

   (a) What do the curves in (a) and (b) above have in common?

   (b) What is special about the curve in (c)?

   (c) What can you say about $\mathbf{F}$? Prove it.

   (d) What is the work done by $\mathbf{F}$ along the connected path consisting of the straight line segment from $(-1,-1)$ to $(1,-1)$ followed by the straight line segment from $(1,-1)$ to $(1,1)$?

   (e) Find a potential function for $\mathbf{F}$.

   (f) Check that we get the answer in (d) by using the potential function found in (e).

   (g) Find the work done by $\mathbf{F}$ along **any** curve that starts at $(1,3)$ and ends at $(2,5)$.

   (h) Find the work done by $\mathbf{F}$ along the curve

$$\mathbf{x}(\mathbf{t}) = \begin{pmatrix} e^t\cos(2\pi t) \\ e^t\sin(2\pi t) \end{pmatrix}, \qquad \mathbf{0} \leq \mathbf{t} \leq \mathbf{1}.$$