The following are solutions to the Math 229 Integration Worksheet - Substitution Method. Here's the link to that worksheet `http://www.math.niu.edu/courses/math229/misc/int_prac.pdf`

1. $\int (5x+4)^5 \, dx$

   (a) Let $u = 5x + 4$

   (b) Then $du = 5 \, dx$ or $\frac{1}{5} \, du = dx$.

   (c) Now substitute

$$\begin{aligned} \int (5x+4)^5 \, dx &= \int u^5 \cdot \frac{1}{5} \, du \\ &= \int \frac{1}{5} u^5 \, du \\ &= \frac{1}{30} u^6 + C \\ &= \frac{1}{30}(5x+4)^6 + C \end{aligned}$$

2. $\int 3t^2(t^3+4)^5 \, dt$

   (a) Let $u = t^3 + 4$

   (b) Then $du = 3t^2 \, dt$

   (c) Now substitute

$$\begin{aligned} \int 3t^2(t^3+4)^5 \, dt &= \int (t^3+4)^5 \cdot 3t^2 \, dt \\ &= \int u^5 \cdot du \\ &= \frac{1}{6} u^6 + C \\ &= \frac{1}{6}(t^3+4)^6 + C \end{aligned}$$

3. $\int \sqrt{4x-5} \, dx$

(a) Let $u = 4x - 5$

(b) Then $du = 4\ dx$ or $\frac{1}{4}\ du = dx$

(c) Now substitute

$$\begin{aligned}
\int \sqrt{4x-5}\ dx &= \int \sqrt{u} \cdot \frac{1}{4}\ du \\
&= \int \frac{1}{4} u^{1/2}\ du \\
&= \frac{1}{4} u^{3/2} \cdot \frac{2}{3} + C \\
&= \frac{1}{6}(4x-5)^{3/2} + C
\end{aligned}$$

4. $\int t^2 (t^3+4)^{-1/2}\ dt$

(a) Let $u = t^3 + 4$

(b) Then $du = 3t^2\ dt$ or $\frac{1}{3}\ du = t^2\ dt$

(c) Now substitute

$$\begin{aligned}
\int t^2(t^3+4)^{-1/2}\ dt &= \int (t^3+4)^{-1/2} \cdot t^2\ dt \\
&= \int u^{-1/2} \cdot \frac{1}{3}\ du \\
&= \int \frac{1}{3} u^{-1/2}\ du \\
&= \frac{1}{3} u^{1/2} \cdot \frac{2}{1} + C \\
&= \frac{2}{3} u^{1/2} + C \\
&= \frac{2}{3}(t^3+4)^{1/2} + C
\end{aligned}$$

5. $\int \cos(2x+1)\ dx$

(a) Let $u = 2x + 1$

(b) Then $du = 2\ dx$ or $\frac{1}{2}\ du = dx$

(c) Now substitute

$$\begin{aligned} \int \cos(2x+1)\,dx &= \int \cos(u)\cdot\frac{1}{2}\,du \\ &= \int \frac{1}{2}\cos(u)\,du \\ &= \frac{1}{2}\sin(u)+C \\ &= \frac{1}{2}\sin(2x+1)+C \end{aligned}$$

6. $\int \sin^{10}(x)\cos(x)\,dx$

(a) Let $u = \sin(x)\,dx$

(b) Then $du = \cos(x)\,dx$

(c) Now substitute

$$\begin{aligned} \int \sin^{10}(x)\cos(x)\,dx &= \int u^{10}\cdot du \\ &= \frac{1}{11}u^{11}+C \\ &= \frac{1}{11}\sin^{11}(x)+C \end{aligned}$$

7. $\int \frac{\sin(x)}{(\cos(x))^5}\,dx$

(a) Let $u = \cos(x)$

(b) Then $du = -\sin(x)\,dx$ or $-du = \sin(x)\,dx$

(c) Now substitute

$$\begin{aligned}
\int \frac{\sin(x)}{(\cos(x))^5}\,dx &= \int \frac{1}{(\cos(x)^5} \cdot \sin(x)\,dx \\
&= \int \frac{1}{u^5}\,(-du) \\
&= \int -u^{-5} + C \\
&= -\frac{u^{-4}}{-4} + C \\
&= u^{-4} + C \\
&= \frac{1}{(\cos(x))^4} + C
\end{aligned}$$

8. $\displaystyle\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}\,dx$

(a) Let $u = \sqrt{x} - 1$

(b) Then $du = \dfrac{1}{2\sqrt{x}}\,dx$ or $2\,du = \dfrac{1}{\sqrt{x}}\,dx$

(c) Now substitute

$$\begin{aligned}
\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}\,dx &= \int (\sqrt{x}-1)^2 \cdot \frac{1}{\sqrt{x}}\,dx \\
&= \int u^2\,(2\,du) \\
&= \int 2u^2\,du \\
&= \frac{2}{3}u^3 + C \\
&= \frac{2}{3}(\sqrt{x}-1)^3 + C
\end{aligned}$$

9. $\displaystyle\int \sqrt{x^3+x^2}\,(3x^2+2x)\,dx$

(a) Let $u = x^3 + x^2$

(b) Then $du = (3x^2+2x)\,dx$

(c) Now substitute

$$\begin{aligned}\int \sqrt{x^3+x^2}\cdot(3x^2+2x)\,dx &= \int \sqrt{u}\,du\\ &= \int u^{1/2}\,du\\ &= \frac{2}{3}u^{3/2}+C\\ &= \frac{2}{3}(x^3+x^2)^{3/2}+C\end{aligned}$$

10. $\displaystyle\int_{-1}^{1}\frac{x+1}{(x^2+2x+2)^3}\,dx$

(a) Let $u = x^2+2x+2$

(b) Then $du = (2x+2)\,dx \to du = 2(x+1)\,dx$ or $\frac{1}{2}\,du = (x+1)\,dx$

(c) If $x=-1$, then $u=(-1)^2+2(-1)+2=1$

(d) If $x=1$, then $u=(1)^2+2(1)+2=5$

(e) Now substitute

$$\begin{aligned}\int_{-1}^{1}\frac{(x+1)}{(x^2+2x+2)^3}\,dx &= \int_{-1}^{1}\frac{1}{(x^2+2x+2)^3}\cdot(x+1)\,dx\\ &= \int_{1}^{5}\frac{1}{u^3}\,\frac{1}{2}\,du\\ &= \int_{1}^{5}\frac{1}{2}u^{-3}\,du\\ &= \frac{1}{2}\frac{u^{-2}}{-2}\bigg|_{1}^{5}\\ &= -\frac{1}{4u^2}\bigg|_{1}^{5}\\ &= \left[-\frac{1}{4(5)^2}\right]-\left[-\frac{1}{4(1)^2}\right]\\ &= -\frac{1}{100}+\frac{1}{4}\\ &= \frac{24}{100}\\ &= \frac{6}{25}\end{aligned}$$

11. $\displaystyle\int_0^{\pi} \cos(x)\sqrt{\sin(x)}\, dx$

(a) Let $u = \sin(x)$

(b) Then $du = \cos(x)\ dx$

(c) If $x = 0$, then $u = \sin(0) = 0$.

(d) If $x = \pi$, then $u = \sin(\pi) = 0$

(e) Now substitute

$$\begin{aligned}
\int_0^{\pi} \cos(x)\sqrt{\sin(x)}\, dx &= \int_0^{\pi} \sqrt{\sin(x)} \cdot \cos(x)\ dx \\
&= \int_0^{0} \sqrt{u}\ du \\
&= \int_0^{0} u^{1/2}\ du \\
&= \left.\frac{2}{3}u^{3/2}\right|_0^0 \\
&= \left[\frac{2}{3}(0)^{3/2}\right] - \left[\frac{2}{3}(0)^{3/2}\right] \\
&= 0
\end{aligned}$$

Note, $\displaystyle\int_a^a f(x)\ dx = 0$. So we didn't actually need to go through the last 5 lines.

12. $\displaystyle\int (x+1)\sin(x^2+2x+3)\ dx$

(a) Let $u = x^2 + 2x + 3$

(b) Then $du = (2x+2)\ dx \rightarrow du = 2(x+1)\ dx$ or we can write $\frac{1}{2}\ du = (x+1)\ dx$

(c) Now substitute

$$\begin{aligned}
\int (x+1)\sin(x^2+2x+3)\ dx &= \int \sin(x^2+2x+3) \cdot (x+1)\ dx \\
&= \int \sin(u) \cdot \frac{1}{2}\ du \\
&= \int \frac{1}{2}\sin(u)\ du \\
&= -\frac{1}{2}\cos(u) + C \\
&= -\frac{1}{2}\cos(x^2+2x+3) + C
\end{aligned}$$

13. $\displaystyle\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\,dt$

(a) Let $u = 1 + \dfrac{1}{t}$

(b) Then $du = -\dfrac{1}{t^2}\,dt$ or $-du = \dfrac{1}{t^2}\,dt$

(c) Now substitute

$$\begin{aligned}\int \left(1+\frac{1}{t}\right)^3 \cdot \frac{1}{t^2}\,dt &= \int u^3\,(-du)\\ &= -\frac{1}{4}u^4 + C\\ &= -\frac{1}{4}\left(1+\frac{1}{t}\right)^4 + C\end{aligned}$$

14. $\displaystyle\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx$

(a) Let $u = x^3 + 1$

(b) Then $du = 3x^2\,dx$ or $\dfrac{1}{3}\,du = x^2\,dx$

(c) If $x = -1$, then $u = (-1)^3 + 1 = 0$.

(d) If $x = 1$, then $u = (1)^3 + 1 = 2$

(e) Now substitute

$$\begin{aligned}\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx &= \int_{-1}^{1}\sqrt{x^3+1}\cdot x^2\,dx\\ &= \int_0^2 \sqrt{u}\cdot\frac{1}{3}\,du\\ &= \int_0^2 \frac{1}{3}u^{1/2}\,du\\ &= \frac{1}{3}u^{3/2}\cdot\frac{2}{3}\bigg|_0^2\\ &= \frac{2}{9}u^{3/2}\bigg|_0^2\\ &= \left[\frac{2}{9}(2)^{3/2}\right] - \left[\frac{2}{9}(0)^{3/2}\right]\\ &= \frac{2}{9}\sqrt{8} = \frac{4\sqrt{2}}{9}\end{aligned}$$

15. $\displaystyle\int \frac{2}{\sqrt{3x-7}}\,dx$

(a) Let $u = 3x - 7$

(b) Then $du = 3\,dx$ or $\frac{1}{3}\,du = dx$

(c) Now substitute

$$\begin{aligned}\int \frac{2}{\sqrt{3x-7}}\,dx &= \int \frac{2}{\sqrt{u}} \cdot \frac{1}{3}\,du \\ &= \int \frac{2}{3}u^{-1/2}\,du \\ &= \frac{2}{3}u^{1/2} \cdot \frac{2}{1} + C \\ &= \frac{4}{3}u^{1/2} + C \\ &= \frac{4}{3}\sqrt{3x-7} + C\end{aligned}$$

16. $\displaystyle\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,dx$

(a) Let $u = \sqrt{x} + 1$

(b) Then $du = \frac{1}{2\sqrt{x}}\,dx$ or $2\,du = \frac{1}{\sqrt{x}}\,dx$

(c) If $x = 1$, then $u = \sqrt{1} + 1 = 2$.

(d) If $x = 4$, then $u = \sqrt{4} + 1 = 3$.

(e) Now substitute

$$\begin{aligned}
\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,dx &= \int_1^4 \frac{1}{(\sqrt{x}+1)^2}\cdot\frac{1}{\sqrt{x}}\,dx \\
&= \int_2^3 \frac{1}{u^2}\cdot(2\,du) \\
&= \int_2^3 2u^{-2}\,du \\
&= 2\frac{u^{-1}}{-1}\bigg|_2^3 \\
&= -\frac{2}{u}\bigg|_2^3 \\
&= \left[-\frac{2}{3}\right]-\left[-\frac{2}{2}\right] \\
&= -\frac{2}{3}+1 \\
&= \frac{1}{3}
\end{aligned}$$

17. $\displaystyle\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$

(a) Let $u = x+1$, then $x = u-1$ (need this for later)

(b) Then $du = dx$

(c) If $x = 0$, then $u = 1$.

(d) If $x = 1$, then $u = 2$

(e) Now substitute

$$\begin{aligned}
\int_0^1 \frac{x}{\sqrt{x+1}}\,dx &= \int_1^2 \frac{u-1}{\sqrt{u}}\,du \\
&= \int_1^2 \frac{u}{u^{1/2}} - \frac{1}{u^{1/2}}\,du \\
&= \int_1^2 u^{1/2} - u^{-1/2}\,du \\
&= \frac{2}{3}u^{3/2} - 2u^{1/2}\Big|_1^2 \\
&= \left[\frac{2}{3}(2)^{3/2} - 2(2)^{1/2}\right] - \left[\frac{2}{3}(1)^{3/2} - 2(1)^{1/2}\right] \\
&= \frac{4\sqrt{2}}{3} - 2\sqrt{2} - \frac{2}{3} + 2 \\
&= -\frac{2\sqrt{2}}{3} + \frac{4}{3} \\
&= \frac{4-2\sqrt{2}}{3}
\end{aligned}$$

18. $\int x\sqrt{2x+1}\,dx$

(a) Let $u = 2x+1$. Then $x = \frac{1}{2}(u-1)$ (need this for later)

(b) Then $du = 2\,dx$ or $\frac{1}{2}\,du = dx$

(c) Now substitute

$$\begin{aligned}
\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\cdot\frac{1}{2}\,du \\
&= \int \frac{1}{4}(u-1)\cdot u^{1/2}\,du \\
&= \int \frac{1}{4}u^{3/2} - \frac{1}{4}u^{1/2}\,du \\
&= \frac{1}{4}u^{5/2}\cdot\frac{2}{5} - \frac{1}{4}u^{3/2}\cdot\frac{2}{3} + C \\
&= \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + C \\
&= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C
\end{aligned}$$

19. $\displaystyle\int \sqrt{x}\sqrt{x\sqrt{x}+1}\,dx$

You should rewrite the integral as

$$\int x^{1/2}\sqrt{x^{3/2}+1}\,dx$$

to help identify u.

(a) Let $u = x^{3/2}+1$

(b) Then $du = \frac{3}{2}x^{1/2}\,dx$ or $\frac{2}{3}\,du = x^{1/2}\,dx$

(c) Now substitute

$$\begin{aligned}
\int x^{1/2}\sqrt{x^{3/2}+1}\,dx &= \int \sqrt{x^{3/2}+1}\cdot x^{1/2}\,dx \\
&= \int \sqrt{u}\cdot\frac{2}{3}\,du \\
&= \int \frac{2}{3}u^{1/2}\,du \\
&= \frac{2}{3}u^{3/2}\cdot\frac{2}{3}+C \\
&= \frac{4}{9}u^{3/2}+C \\
&= \frac{4}{9}(x^{3/2}+1)^{3/2}+C
\end{aligned}$$

20. $\displaystyle\int x^3\sqrt{x^2+1}\,dx$

(a) Let $u = x^2-1$. Then $x^2 = u-1$ (need this for later)

(b) Then $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$

(c) Now substitute

$$\begin{aligned}
\int x^3\sqrt{x^2+1}\,dx &= \int x^2\sqrt{x^2+1}\cdot x\,dx \\
&= \int (u-1)\sqrt{u}\,\frac{1}{2}\,du \\
&= \int \frac{1}{2}u^{3/2} - \frac{1}{2}u^{1/2}\,du \\
&= \frac{1}{2}u^{5/2}\cdot\frac{2}{5} - \frac{1}{2}u^{3/2}\cdot\frac{2}{3} + C \\
&= \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + C \\
&= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C
\end{aligned}$$

21. $\int (x^2+1)\sqrt{x-2}\,dx$

(a) Let $u = x-2$. Then $u+2 = x$ and $(u+2)^2 = x^2$

$$x^2 = u^2 + 4u + 4$$

(b) Then $du = dx$

(c) Now substitute

$$\begin{aligned}
\int (x^2+1)\sqrt{x-2}\,dx &= \int (u^2+4u+4+1)\sqrt{u}\,du \\
&= \int (u^2+4u+5)\sqrt{u}\,du \\
&= \int u^{5/2} + 4u^{3/2} + 5u^{1/2}\,du \\
&= \frac{2}{7}u^{7/2} + \frac{8}{5}u^{5/2} + \frac{10}{3}u^{3/2} + C \\
&= \frac{2}{7}(x-2)^{7/2} + \frac{8}{5}(x-2)^{5/2} + \frac{10}{3}(x-2)^{3/2} + C
\end{aligned}$$

22. $\int \frac{x^2+2x}{x^2+2x+1}\,dx$

(a) First, let's simplify by doing some long division. The work is omitted as I don't know how to tex long division.

$$\frac{x^2+2x}{x^2+2x+1} = 1 - \frac{1}{x^2+2x+1} = 1 - \frac{1}{(x+1)^2}$$

(b) So we are evaluating $\int 1 - \frac{1}{(x+1)^2}\, dx$

$$\int 1 - \frac{1}{(x+1)^2}\, dx \quad = \quad x + \frac{1}{x+1} + C$$

23. $\int \frac{1}{x^2+6x+9}$

(a) First, rewrite the integrand as $\frac{1}{x^2+6x+9} = \frac{1}{(x+3)^2}$

(b) Rewrite the integral

$$\int \frac{1}{(x+3)^2}\, dx$$

(c) Let $u = x+3$

(d) Then $du = dx$

$$\begin{aligned} \int \frac{1}{(x+3)^2}\, dx &= \int \frac{1}{u^2}\, du \\ &= \int u^{-2}\, du \\ &= -u^{-1} + C \\ &= -(x+3)^{-1} + C \\ &= -\frac{1}{x+3} + C \end{aligned}$$

24. $\int \frac{\sec^2(x)}{(1+\tan(x))^3}\, dx$

(a) Let $u = 1 + \tan(x)$

(b) Then $du = \sec^2(x)\, dx$

(c) Now substitute

$$\begin{aligned}
\int \frac{\sec^2(x)}{(1+\tan(x))^3}\,dx &= \int \frac{1}{(1+\tan(x))^3}\cdot \sec^2(x)\,dx \\
&= \int \frac{1}{u^3}\,du \\
&= \int u^{-3}\,du \\
&= -\frac{1}{2}u^{-2} + C \\
&= -\frac{1}{2u^2} + C \\
&= -\frac{1}{2(1+\tan(x))^2} + C
\end{aligned}$$

25. $\displaystyle\int \frac{\sin(x)}{(2+3\cos(x))^2}\,dx$

(a) Let $u = 2 + 3\cos(x)$

(b) Then $du = -3\sin(x)\,dx$ or $-\dfrac{1}{3}\,du = \sin(x)\,dx$

(c) Now substitute

$$\begin{aligned}
\int \frac{\sin(x)}{(2+3\cos(x))^2}\,dx &= \int \frac{1}{(2+3\cos(x))^2}\cdot \sin(x)\,dx \\
&= \int \frac{1}{u^2}\cdot -\frac{1}{3}\,du \\
&= \int -\frac{1}{3}u^{-2}\,du \\
&= \frac{1}{3}u^{-1} + C \\
&= \frac{1}{3}(2+3\cos(x))^{-1} + C
\end{aligned}$$

26. $\displaystyle\int x\tan(x^2)\sec(x^2)\,dx$

(a) Let $u = x^2$

(b) Then $du = 2x\,dx$ or $\dfrac{1}{2}\,du = x\,dx$

(c) Now substitute

$$\begin{aligned}
\int x\tan(x^2)\sec(x^2)\,dx &= \int \tan(x^2)\sec(x^2)\cdot x\,dx \\
&= \int \tan(u)\sec(u)\cdot\frac{1}{2}\,du \\
&= \int \frac{1}{2}\tan(u)\sec(u)\,du \\
&= \frac{1}{2}\sec(u)+C \\
&= \frac{1}{2}\sec(x^2)+C
\end{aligned}$$

27. $\int 3t^3(t^2+4)^5\,dt$

(a) Let $u=t^2+4$. Then $t^2=u-4$ (need this later)

(b) Then $du=2t\,dt$ or $\frac{1}{2}\,du=t\,dt$

(c) Now substitute

$$\begin{aligned}
\int 3t^3(t^2+4)^5\,dt &= \int 3t^2(t^2+4)^5\cdot t\,dt \\
&= \int 3(u-4)(u)^5\cdot\frac{1}{2}\,du \\
&= \int \frac{3}{2}(u^6-4u^5)\,du \\
&= \frac{3}{2}\left(\frac{1}{7}u^7-\frac{4}{6}u^6\right)+C \\
&= \frac{3}{14}u^7-u^6+C \\
&= \frac{3}{14}(t^2+4)^7-(t^2+4)^6+C
\end{aligned}$$

28. $\int (\tan(2x)+\cot(2x))^2\,dx$

(a) Since there isn't an obvious substitution, let's foil and see what happens. This will require some trig identities.

$$\begin{aligned}
(\tan(2x)+\cot(2x))^2 &= (\tan(2x)+\cot(2x))\cdot(\tan(2x)+\cot(2x)) \\
&= \tan^2(2x)+2\tan(2x)\cot(2x)+\cot^2(2x) \\
&= \tan^2(2x)+2+\cot^2(2x) \\
&= (\sec^2(2x)-1)+2+(\csc^2(2x)-1) \\
&= \sec^2(2x)+\csc^2(2x)
\end{aligned}$$

(b) The reason we used the trig identities at the end is because we know nice functions whose derivatives are $\sec^2()$ and $\csc^2()$.

(c) Let's rewrite the integral

$$\int \sec^2(2x)+\csc^2(2x)\;dx$$

(d) Let $u = 2x$

(e) Then $du = 2\;dx$ or $\frac{1}{2}\;du = dx$

(f) Now substitute

$$\begin{aligned}
\int \sec^2(2x)+\csc^2(2x)\;dx &= \int(\sec^2(u)+\csc^2(u))\cdot\frac{1}{2}\;du \\
&= \frac{1}{2}\int \sec^2(u)+\csc^2(u)\;du \\
&= \frac{1}{2}(\tan(u)-\cot(u))+C \\
&= \frac{1}{2}(\tan(2x)-\cot(2x))+C \\
&= \frac{1}{2}\tan(2x)-\frac{1}{2}\cot(2x)+C
\end{aligned}$$