Error of Taylor Polynomials

1. Find a bound on $|R_2 f(x)|$ for $f(x) = x\sin^3 x$ for $-1 \le x \le 1$.

$$f'''(x) = -9\sin^3 x - 21x\cos x\sin^2 x + 18\cos^2 x\sin x + 6x\cos^3 x,$$

so by the triangle inequality, we have

$$|f'''(x)| \le 9|\sin^3 x| + 21|\cos|\sin^2 x + 18\cos^2 x|\sin x| + 6|x||\cos x|^3$$

Since $-1 \le x \le 1$, $|x| \le 1$, and $|\sin x| \le 1$ and $|\cos x| \le 1$. Therefore, for all x between -1 and 1,

$$|f'''(x)| \le 9 + 21 + 18 + 6 = 54 = M$$

So

$$\begin{aligned} |R_3 f(x)| &\le \frac{M}{3!} \cdot x^3 \\ &\le \frac{54}{3!} \cdot 1^3 \\ &\le 9 \end{aligned}$$

2. Find B so that $|R_4 \cos x| \le B$ for all $0 \le x \le 2\pi$.

Let $f(x) = \cos x$. Then $f^{(5)}(x) = -\sin x$, and $|f^{(4)}(x)| \le 1 = M$ for all x. So,

$$\begin{aligned} |R_4 \cos x| &\le \frac{M}{5!} \cdot x^5 \\ &\le \frac{1}{5!} \cdot (2\pi)^5 \end{aligned}$$

Thus, $B = \frac{1}{5!} \cdot (2\pi)^5$.

3. Find B so that $|R_{100} e^x| \le B$ for all $-10 \le x \le 10$.

Let $f(x) = e^x$. Then $f^{(101)}(x) = e^x$, and when $-10 \le x \le 10$, $e^x \le e^{10} = M$. Therefore,

$$\begin{aligned} |R_{100} e^x| &\le \frac{M}{101!} \cdot x^{101} \\ &\le \frac{e^{10}}{101!} \cdot (10)^{101} \end{aligned}$$

Thus, $B = \frac{e^{10}}{101!} \cdot (10)^{101}$.

4. Find n so that $|T_n e^x - e^x| \le 0.01$ for $-1 \le x \le 1$.

Let $f(x) = e^x$. Then $|T_n e^x - e^x| = |R_n e^x| \le \frac{M}{(n+1)!} \cdot x^{n+1}$. $f^{(n)}(x) = e^x$, and for $-1 \le x \le 1$, $|f^{(n)}(x)| = |e^x| \le e^1 = M$. Thus, since our desired error bound is 0.01, we want to find n so that $\frac{e}{(n+1)!} \cdot 1^{n+1} = \frac{e}{(n+1)!} \le 0.01$. We test values of n until we find that $n = 5$ gives $\frac{3}{820} \le \frac{1}{100}$, so $n = 5$ works. In fact, any $n \ge 5$ is a correct solution.

5. Approximate $\cos(0.1)$ using $T_5 \cos x$ and find a bound on the error.

$$T_5 \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

so plugging in $x = 0.1$ yields $\cos(0.1) \simeq 1 - \frac{0.1^2}{2!} + \frac{0.1^4}{4!}$.

To find the error, we need to bound $|R_5 \cos x|$. Since 0.1 is between 0 and 1, we can consider only the interval $0 \leq x \leq 1$. If $f(x) = \cos x$, then $f^{(6)}(x) = -\cos x$, and for all x, $|f^{(6)}(x)| \leq 1 = M$. Thus

$$\begin{aligned} |R_5 \cos x| &\leq \frac{M}{6!} \cdot x^6 \\ &\leq \frac{1}{6!} \cdot 1^6 \\ &\leq \frac{1}{6!} \end{aligned}$$

6. Find n so that $T_n e^x$ approximates $\sqrt[3]{e}$ to four decimal places.

To be correct to 4 decimal places, we want the error, $|R_n e^x|$ to be less than $\frac{1}{10^4}$. Since we are interested in $x = 1/3$, we can consider only the interval $0 \leq x \leq 1$. Let $f(x) = e^x$. Since $f^{(n)}(x) = e^x$, and $|f^{(n)}(x)| \leq e^1 = M$ on $0 \leq x \leq 1$, we have

$$\begin{aligned} |R_n f(x)| &\leq \frac{M}{(n+1)!} \cdot x^{n+1} \\ &\leq \frac{e}{(n+1)!} \cdot 1^{n+1} \\ &\leq \frac{e}{(n+1)!} \end{aligned}$$

Testing different values of n, we find that $n = 7$ gives $|R_n f(x)| \leq \frac{e}{8!} = \frac{e}{40320} \leq \frac{1}{10000}$, since $e \leq 3$.