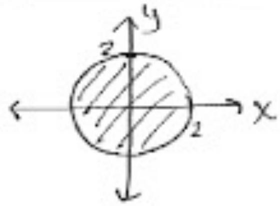
Worksheet 10 Solutions

① (a) $4 - x^2 - y^2 \geq 0$

$x^2 + y^2 \leq 4$

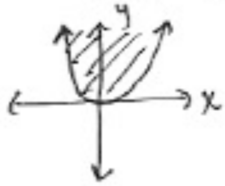


(b) $4 + x^2 + y^2 \geq 0$

$x^2 + y^2 \geq -4$. This is always true, so the domain is all real numbers.

(c) $y - x^2 \geq 0$

$y \geq x^2$



② (a) γ_1: $\begin{pmatrix} 1+t^2 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow t = -1$, but $1 + (-1)^2 \neq 0$, so no.

γ_2: $\begin{pmatrix} 1+t^3 \\ t^3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow t = -1$: $\begin{matrix} 1 + (-1)^3 = 0 \\ (-1)^3 = -1 \end{matrix}$, so yes.

γ_3: $\begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow$ $2\cos t = 0 \Rightarrow t = \pi/2 + n\pi$, so no.

$2\sin(\pi/2 + n\pi) = \pm 2 \neq -1$

(b) γ_2 traces out a line, γ_3 is a circle, while γ_1 is a parabola.

(c) γ_1: $x = 1 + y^2$, γ_2: $x = 1 + y$, γ_3: $x^2 + y^2 = 4$.

③ $A(1,3,1)$ $B(2,2,-1)$ $C(2,0,3)$

(a) $\vec{AB} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ $\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & -1 & -2 \\ 1 & -3 & 2 \end{vmatrix} = i(-2-6) - j(2+2) + k(-3+1)$

$= -8i - 4j - 2k$

$\vec{n} = \begin{pmatrix} -8 \\ -4 \\ -2 \end{pmatrix}$

Plane: $\vec{n} \cdot \vec{AX} = 0$

$\begin{pmatrix} -8 \\ -4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y+1 \\ z+2 \end{pmatrix} = 0 \quad \rightsquigarrow \quad -8(x-1) - 4(y+1) - 2(z+2) = 0$

(b) Area of $\Delta ABC = \frac{1}{2}$(area of parallelogram spanned by $\vec{AB}$ & $\vec{AC}$)

$= \frac{1}{2} \cdot \| \vec{AB} \times \vec{AC} \|$

$= \frac{1}{2} \cdot \sqrt{64 + 16 + 4}$

$= \frac{1}{2}\sqrt{84}$

(4) (a) $f(x,y)=x^2y, \quad \gamma(t)=\begin{pmatrix} t^2 \\ t \end{pmatrix}, \; t=1$

$(f\circ\gamma)(t) = (t^2)^2 \cdot t = t^5$

$(f\circ\gamma)'(t) = 5t^4$

$(f\circ\gamma)'(1) = 5$

(b) $f(x,y,z) = x+yz, \quad \gamma(t) = \begin{pmatrix} t \\ 2t \\ 3t \end{pmatrix}$

$t=2.$

$(f\circ\gamma)(t) = t+(2t)(3t)$

$= t+6t^2$

$(f\circ\gamma)'(t) = 1+12t$

$(f\circ\gamma)'(2) = 1+24 = 25$

(5) $\gamma(t) = \begin{pmatrix} \cos(2t) \\ \sin(2t) \\ 4\sqrt{2}\, t \end{pmatrix}$

(a) $\begin{pmatrix} \cos(2t) \\ \sin(2t) \\ 4\sqrt{2}\, t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \pi\sqrt{2} \end{pmatrix} \rightsquigarrow \quad 4\sqrt{2}\,t = \pi\sqrt{2}$

$t = \pi/4$

$\gamma'(t) = \begin{pmatrix} -2\sin(2t) \\ 2\cos(2t) \\ 4\sqrt{2} \end{pmatrix}, \quad \gamma'(\pi/4) = \begin{pmatrix} -2 \\ 0 \\ 4\sqrt{2} \end{pmatrix}$, so $\ell(s) = \begin{pmatrix} 0 \\ 1 \\ \pi\sqrt{2} \end{pmatrix} + s\begin{pmatrix} -2 \\ 0 \\ 4\sqrt{2} \end{pmatrix}$

(b) In the xy-plane, $z=0$, so $\pi\sqrt{2} + 4\sqrt{2}\cdot s = 0$

$4\sqrt{2}\,s = -\pi\sqrt{2}$

$s = -\pi/4$

$\ell(-\pi/4) = \begin{pmatrix} \pi/2 \\ 1 \\ 0 \end{pmatrix}$

$(\pi/2, 1)$, yes.

(c) $\|\gamma'(t)\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 32} = \sqrt{36} = 6 \leftarrow$ speed $= s$

$\gamma''(t) = \begin{pmatrix} -4\cos(2t) \\ -4\sin(2t) \\ 0 \end{pmatrix}, \quad \gamma''(\pi/4) = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} \leftarrow$ acceleration.

(d) linear accel $= s' = 0$

angular accel.: $s\|T'\|$

$T = \frac{1}{6}\begin{pmatrix} -2\sin(2t) \\ 2\cos(2t) \\ 4\sqrt{2} \end{pmatrix} \rightsquigarrow \quad T' = \frac{1}{6}\begin{pmatrix} -4\cos(2t) \\ -4\sin(2t) \\ 0 \end{pmatrix}$

$\|T'\| = \frac{1}{6}\sqrt{16\cos^2(2t) + 16\sin^2(2t)} = \frac{1}{6}\cdot 4$

$s\|T'\| = 6\cdot\frac{1}{6}\cdot 4 = 4$

Alternatively, one could realize that if one of these 2 components is 0, then the other component is the magnitude of acceleration.

⑥ (a) False - acceleration also changes the direction of the velocity.

(b) True: If $\|v\| = k$

$\|v\|^2 = k^2$, so $v \cdot v = k^2$. Differentiate both sides:

$a \cdot v + v \cdot a = 0 \Rightarrow 2v \cdot a = 0 \Rightarrow v \cdot a = 0 \Rightarrow v \perp a$.

(c) False: only if velocity is constant.

(d) True: If $v \perp a$, then the linear acceleration is 0, so the speed is not changing.

Alternate explanation of (b): If speed is constant, then the linear acceleration is zero, so the acceleration is all angular acceleration, which is $\perp$ to v.

⑦ (a) $z^2 \overset{?}{=} x^2 + y^2$.

u: $(t\cos t)^2 + (t\sin t)^2 \overset{?}{=} (t^2)^2$

$t^2\cos^2 t + t^2\sin^2 t \overset{?}{=} t^4$

$t^2 \overset{?}{=} t^4$

False, so u doesn't lie on this surface.

v: $(\cos t)^2 + (2\sin t)^2 \overset{?}{=} (-1-3\sin^2 t)^2$

$\cos^2 t + 4\sin^2 t \overset{?}{=} 1 + 6\sin^2 t + 9\sin^4 t$

$\cos^2 t \overset{?}{=} 1 + 2\sin^2 t + 9\sin^4 t$

$1 - \sin^2 t \overset{?}{=} 1 + 2\sin^2 t + 9\sin^4 t$

$0 \overset{?}{=} 3\sin^2 t + 9\sin^4 t$

False (i.e, not true for all values of t - try $t = \pi/2$), so no.

w: $(e^t\cos(5t))^2 + (e^t\sin(5t))^2 \overset{?}{=} (e^t)^2$

$e^{2t}\cos^2(5t) + e^{2t}\sin^2(5t) \overset{?}{=} e^{2t}$

$e^{2t} = e^{2t}$ ✓

True, so w lies on this surface.

(b) $z = -x^2 - y^2$

u: $t^2 \overset{?}{=} -(t\cos t)^2 - (t\sin t)^2$

$\overset{?}{=} -t^2(\cos^2 t + \sin^2 t)$

$\overset{?}{=} -t^2$

No.

v: $-1-3\sin^2 t \overset{?}{=} -\cos^2 t - 4\sin^2 t$

$1 + 3\sin^2 t \overset{?}{=} \cos^2 t + \sin^2 t + 3\sin^2 t$

$= 1 + 3\sin^2 t$ ✓

True, so v lies in this surface.

w: $e^t \overset{?}{=} -(e^t\cos(5t))^2 - (e^t\sin(5t))^2$

$\overset{?}{=} -e^{2t}(\cos^2(5t) + \sin^2(5t))$

$\overset{?}{=} -e^{2t}$

False, so no.

⑧ $f(x,y) = 2x^3 - \frac{1}{4}xy^2$

(a) $(1, 2, f(1,2))$

$f_x(x,y) = 6x^2 - \frac{1}{4}y^2 \rightarrow f_x(1,2) = 6 - 1 = 5$

$f_y(x,y) = -\frac{1}{2}xy \longrightarrow f_y(1,2) = -1$

$f(1,2) = 2 - \frac{1}{4}\cdot 1\cdot 4 = 2 - 1 = 1$

$z = 1 + 5(x-1) - 1(y-2)$

(b) $f(0.95, 2.1) \approx 1 + 5(0.95-1) - (2.1-2)$

$\approx 1 + 5(-0.05) - 0.1$

$\approx 1 + -0.25 - 0.1$

$\approx 1 - 0.35$

≈ 0.65

⑨ $\|u\| = 5$, $\|w\| = 2$, $u\cdot v = 7$, $u\cdot w = -3$.

(a) $u\cdot u = \|u\|^2 = 25$

(b) $w\cdot 2u = 2(u\cdot w) = 2(-3) = -6$

(c) $u\cdot(v-w) = u\cdot v - u\cdot w = 7 - (-3) = 10$

(d) $(u+w)\cdot(u-w) = u\cdot u \cancel{- u\cdot w + u\cdot w} - w\cdot w$

$= \cancel{25 - 7 - 3}$

$= u\cdot u - w\cdot w = \|u\|^2 - \|w\|^2$

$= 25 - (2)^2$

$= 21$