Worksheet 14 Solutions

② (d) $f(x,y) = x^4 - y^2(1+x^2)$ on $x^2+y^2 \le 16$.

(x,y)	$f(x,y)$
$(0,0)$	0
$(0,4)$	-16
$(0,-4)$	-16
$\left(\frac{15}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$	$-\frac{706}{16} < -16$
$\left(\frac{-15}{\sqrt{2}}, \frac{-7}{\sqrt{2}}\right)$	
$\left(\frac{-15}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$	
$\left(\frac{15}{\sqrt{2}}, \frac{-7}{\sqrt{2}}\right)$	
$(\pm 4, 0)$	$4^4 > 0$

Interior: $\nabla f = 0$

$$\nabla f = \begin{pmatrix} 4x^3 - 2xy^2 \\ -2y - 2x^2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$4x^3 - 2xy^2 = 0$

$-2y - 2x^2y = 0 \longrightarrow -2y(1+x^2) = 0$

$y = 0$ or $x^2 + 1 \neq 0$

$\downarrow$

$4x^3 = 0$

$x = 0$

Boundary: $x^2+y^2 = 16$

$y^2 = 16 - x^2$, $-4 \le x \le 4$ ← just as in single-variable calculus, we need to add these endpts to our list.

So we have $f(x, \pm\sqrt{16-x^2}) = x^4 - (16-x^2)(1+x^2) = \underbrace{g(x)}_{\text{just to make the notation nicer.}}$

$g(x) = 2x^4 - 15x^2 - 16$

$g'(x) = 8x^3 - 30x = 0$

$x(8x^2 - 30) = 0$

$x = 0$ or $x^2 = 30/8 = 15/4$

$x = \pm\sqrt{15}/2$

$y^2 = 16 - x^2$:

If $x = 0$, $y^2 = 16$, $y = \pm 4$. If $x = \pm\sqrt{15}/2$, $y^2 = 16 - 15/4 = 49/4$, $y = \pm 7/2$

$$\boxed{\begin{array}{l} \text{Global min @ } \left(\pm\frac{\sqrt{15}}{2}, \pm\frac{7}{\sqrt{2}}\right) \\ \text{Global max @ } (\pm 4, 0) \end{array}}$$