MATH 222 (Lectures 1,2,4) **Worksheet 10** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Hasdrubal has designed a rocket. While proving mathematically that it won't explode, he used the approximation $e^{\frac{1}{3}} \approx 1 + \frac{1}{3} + \frac{1}{3^2 2!} + \frac{1}{3^3(3!)}$. If this approximation is off by more than $\frac{2}{4!}(\frac{1}{3})^4$ then the rocket might blow up. Convince Hasdrubal that it won't.

   Solution:

   We know that if $f(x) = e^x$ then $f^{(n)}(x) = e^x$. Approximating $e^{\frac{1}{3}}$ by $1 + \frac{1}{3} + \frac{1}{3^2 2!} + \frac{1}{3^3(3!)}$ corresponds to approximating e^x by $1+x+\frac{x^2}{2!}+\frac{x^3}{3!} = f(0)+f'(0)x+f^{(2)}(0)\frac{x^2}{2!}+f^{(3)}(0)\frac{x^3}{3!}$. Taylor's theorem then tells us that there is a ξ with $0 \leqslant \xi \leqslant \frac{1}{3}$

   $$e^{\frac{1}{3}} - (1 + \frac{1}{3} + \frac{1}{3^2 2!} + \frac{1}{3^3(3!)}) = \frac{e^{\xi}}{4!}(\frac{1}{3})^4$$

   Since e^x is an increasing function, we can bound this error by

   $$\frac{e^{\xi}}{4!}(\frac{1}{3})^4 \leqslant \frac{e^{\frac{1}{3}}}{4!}(\frac{1}{3})^4$$

   but this is actually not at all helpful. The whole point of this problem was, after all, to estimate $e^{\frac{1}{3}}$! We still know that $e \leqslant 3 < 8$, so that $e^{\frac{1}{3}} < 8^{\frac{1}{3}} = 2$. A final answer is then that

   $$e^{\frac{1}{3}} - (1 + \frac{1}{3} + \frac{1}{3^2 2!} + \frac{1}{3^3(3!)}) < \frac{2}{4!}(\frac{1}{3})^4$$

2. Find a bound for $R_n^0 \sin(3x)$ and use this to show that $T_n^0 \sin(3x) \to \sin(3x)$ for all x as $n \to \infty$.

   Solution: We begin by recalling the Lagrange form of the Taylor remainder term. If $f(x) = \sin(3x)$ then

   $$R_n^0 \sin(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

   Notice that $f^{(n+1)}(x)$ is one of $3^{n+1}\sin(3x), 3^{n+1}\cos(3x), -3^{n+1}\sin(3x)$, or $-3^{n+1}\cos(3x)$. In any of these cases, we know that $|f^{(n+1)}(x)| \leqslant 3^{n+1}$ for all x. So in particular, we have the bound

   $$\begin{aligned} |R_n^0 \sin(3x)| &= |\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}| \\ &\leqslant \frac{3^n}{(n+1)!}|x|^{n+1} \\ &= \frac{|3x|^{n+1}}{(n+1)!} \end{aligned}$$

Since for any x, $\lim_{n\to\infty} \frac{1}{(n+1)!}|3x|^{n+1} = 0$, this shows that for any x, $R_n^0 \sin(3x) \to 0$, which implies that $T_n^0 \sin(3x) = \sin(3x) - R_n^0 \sin(3x) \to \sin(3x)$.

3. Find a bound on $|R_n \cos(x)|_{x=1}|$ and use this information to find a decimal approximation of $\cos(1)$ with an error of at most .1.

   Solution: Recall that $\cos(x) - T_n \cos(x) = R_n \cos(x)$ by definition, so if $|R_n \cos(x)_{x=1}|$ is less than $\frac{1}{10}$, then $\cos(1)$ is within two decimal digits of $T_n \cos(x)|_{x=1}$. If $f(x) = \cos(x)$ then $f^{n+1}(x)$ is $\pm\cos(x)$ or $\pm\sin(x)$. In any case, we have $|f^{n+1}(x)| \leqslant 1$. It follows then that

$$|R_n \cos(x)|_{x=1}| \leqslant \frac{1(1)^{n+1}}{(n+1)!} = \frac{1}{(n+1)!}$$

   so if n is sufficiently large that $\frac{1}{(n+1)!} \leqslant \frac{1}{10}$, then $|R_n \cos(x)|_{x=1}| \leqslant \frac{1}{10}$. This is true, for example, for $n = 3$, since $(3+1)! = 24$. Our approximation is then $T_3 \cos(x)|_{x=1} = 1 - \frac{1}{2} = .5$.

4. Find the Taylor series around zero for $\cosh(2x) = \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$.

   Solution:

$$\begin{aligned}
\frac{1}{2}\left(e^{2x} + e^{-2x}\right) &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} + \frac{(-1)^n 2^n x^n}{n!}\right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!} 2^n x^n
\end{aligned}$$

   We can observe that $1 + (-1)^n = 0$ if n is odd and 2 if n is even. We therefore only need to sum over the even positive integers $n = 2k$

$$\begin{aligned}
&= \frac{1}{2}\sum_{k=0}^{\infty} \frac{2}{(2k)!} 2^{2k} x^{2k} \\
&= \sum_{k=0}^{\infty} \frac{1}{(2k)!} 2^{2k} x^{2k}
\end{aligned}$$

5. Find the degree two Taylor polynomial around 0 of $\frac{e^x}{1-x}$ without computing any derivatives.

Solution: Recall that $e^x = 1 + x + \frac{x^2}{2} + o(x^2)$ and $\frac{1}{1-x} = 1 + x + x^2 + o(x^2)$. Then

$$\begin{aligned}
\frac{e^x}{1-x} &= \left(1 + x + \frac{x^2}{2} + o(x^2)\right)\left(1 + x + x^2 + o(x^2)\right) \\
&= \underbrace{1\left(1 + x + x^2 + o(x^2)\right)}_{\text{term 1}} + \underbrace{x\left(1 + x + x^2 + o(x^2)\right)}_{\text{term 2}} \\
&+ \underbrace{\frac{x^2}{2}\left(1 + x + x^2 + o(x^2)\right)}_{\text{term 3}} + \underbrace{o(x^2)\left(1 + x + x^2 + o(x^2)\right)}_{\text{term 4}} \\
&= \underbrace{1 + x + x^2 + o(x^2)}_{\text{term 1}} + \underbrace{x + x^2 + o(x^2)}_{\text{term 2}} + \underbrace{\frac{x^2}{2} + o(x^2)}_{\text{term 3}} + \underbrace{o(x^2)}_{\text{term 4}} \\
&= 1 + 2x + \frac{5}{2}x^2 + o(x^2)
\end{aligned}$$

Therefore $T_2^0 \frac{e^x}{1-x} = 1 + 2x + \frac{5}{2}x^2$.

6. Let $f(x) = xe^{3x^2}$. Compute $f^{(2015)}(0)$ and $f^{(2016)}(0)$.

   Solution: We start by computing the Taylor series:

$$\begin{aligned}
f(x) &= x^4 e^{3x^2} \\
&= x^4 \sum_{n=0}^{\infty} \frac{1}{n!}(3x^2)^n \\
&= x^4 \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{2n} \\
&= \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{2n+4}
\end{aligned}$$

The coefficient of x^m in the Taylor series is $\frac{1}{m!}f^{(m)}(0)$. Only even powers of x are showing up, so the coefficient $f^{(2015)}(0) = 0$. As for $f^{(2016)}(0)$, the term with x^{2016} is the term with $n = 1006$. Thus

$$\begin{aligned}
\frac{1}{2016!} f^{(2016)}(0) &= \frac{3^{1006}}{1006!} \\
f^{(2016)}(0) &= \frac{3^{1006}}{1006!} 2016!
\end{aligned}$$