MATH 222 (Lectures 1,2,4) **Worksheet 12** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. (a) Find

$$\lim_{n\to\infty} \frac{n^2+n+1}{3n^2-n-2}$$

(b) Find an example of a sequence a_n which is bounded but not convergent.

Solution:

(a)

$$\begin{aligned}\lim_{n\to\infty} \frac{n^2+n+1}{3n^2-n-2} &= \lim_{n\to\infty} \left(\frac{n^2}{n^2}\right) \frac{1+\frac{1}{n}+\frac{1}{n^2}}{3-\frac{1}{n}-\frac{2}{n^2}} \\ &= \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{3-\frac{1}{n}-\frac{2}{n^2}} \\ &= \frac{1}{3}\end{aligned}$$

(b) An example is given by $a_n = (-1)^n$.

2. If $x > 2$, use the geometric series formula to find $\sum_{n=0}^{\infty} \frac{2^{n+1}}{x^n}$.

Solution:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2^{n+1}}{x^n} &= 2\sum_{n=0}^{\infty} \frac{2^n}{x^n} \\ &= 2\sum_{n=0}^{\infty} \left(\frac{2}{x}\right)^n \\ &= \frac{2}{1-\frac{2}{x}}\end{aligned}$$

3. Let $a_n = \frac{1}{n^2-n}$ and $S_N = \sum_{k=2}^{N} a_n$.

(a) Use partial fractions to rewrite a_n.

(b) Use part (a) to write out S_2, S_3 and S_4 explicitly and notice how terms cancel. Generalize this to find a formula for S_N.

(c) Compute $\sum_{k=2}^{\infty} a_n$ $(= \lim_{N\to\infty} S_N)$.

Solution:

(a)

$$\frac{1}{n^2-n} = \frac{A}{n-1} + \frac{B}{n}$$
$$1 = An + B(n-1)$$

so that $A = 1$ and $B = -1$ and $\frac{1}{n^2-n} = \frac{1}{n-1} - \frac{1}{n}$

(b)

$$S_2 = \underbrace{1 - \frac{1}{2}}_{a_2}$$
$$S_3 = \underbrace{1 - \frac{1}{2}}_{a_2} + \underbrace{\frac{1}{2} - \frac{1}{3}}_{a_3}$$
$$= 1 - \frac{1}{3}$$
$$S_4 = \underbrace{1 - \frac{1}{2}}_{a_2} + \underbrace{\frac{1}{2} - \frac{1}{3}}_{a_3} + \underbrace{\frac{1}{3} - \frac{1}{4}}_{a_4}$$
$$= 1 - \frac{1}{4}$$

in general $S_N = 1 - \frac{1}{N}$.

(c) $\lim_{N\to\infty} S_N = 1$.

4. Compute

$$\lim_{n\to\infty} \frac{n!}{n^n}.$$

Solution: Since $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ and $n^n = n \cdot n \cdots n \cdot n$, we have that

$$\begin{aligned} \frac{n!}{n^n} &= \frac{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}{n \cdot n \cdot n \cdots n \cdot n} \\ &= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n} \cdot \frac{1}{n} \end{aligned}$$

Since $\frac{n}{n}, \ldots, \frac{2}{n} \leqslant 1$, it follows that $\frac{n!}{n^n} \leqslant \frac{1}{n}$. Thus,

$$0 \leqslant \frac{n!}{n} \leqslant \frac{1}{n},$$

and so by the Sandwich Theorem and the fact that $\lim_{n\to\infty} \frac{1}{n} = 0$, it is also true that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

5. Compute

$$\lim_{n\to\infty} \frac{n^3+7}{5n^3+\ln n}.$$

Solution: Intuitively, one should think that $n^3 \gg \ln n$, we can think of the denominator as simply $5n^3$, and so

$$\lim_{n\to\infty} \frac{n^3+7}{5n^3+\ln n} = \lim_{n\to\infty} \frac{n^3+7}{5n^3} = \frac{1}{5}.$$

More formally, we can divide the numerator and denominator by n^3 to get:

$$\lim_{n\to\infty} \frac{n^3+7}{5n^3+\ln n} = \lim_{n\to\infty} \frac{1+7/n^3}{5+\frac{\ln n}{n^3}} = \frac{1+0}{5+0} = \frac{1}{5},$$

because $\lim_{n\to\infty} \frac{7}{n^3} = 0$ and $\lim_{n\to\infty} \frac{\ln n}{n^3} = 0$.

6. Compute

$$\lim_{n\to\infty} \frac{8n^3+3\sqrt{n}-\ln n+5^n}{n!}.$$

Solution: We can break this fraction into pieces:

$$\begin{aligned} \lim_{n\to\infty} \frac{8n^3+3\sqrt{n}-\ln n+5^n}{n!} &= \lim_{n\to\infty} \frac{8n^3}{n!} + \lim_{n\to\infty} \frac{3\sqrt{n}}{n!} - \lim_{n\to\infty} \frac{\ln n}{n!} + \lim_{n\to\infty} \frac{5^n}{n!} \\ &= 0+0-0+0 \end{aligned}$$

because factorials grow faster than polynomials, square roots, natural logs, and exponentials.