MATH 222 (Lectures 1,2,4) **Worksheet 14** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Find a bound on $R_1 \sin(x)$ which is valid for all x with $0 \leqslant x \leqslant 1$. Use this to show that there is a constant $C > 0$ so that for all integers $n \geqslant 1$, $\sin\left(\frac{1}{n}\right) \geqslant \frac{1}{n} - \frac{C}{n^2}$. Why does this show that the sum

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges?

Solution: Set $f(x) = \sin(x)$ and notice that $T_1 \sin(x) = x$. To find a bound on $R_1 \sin(x)$, we compute $f''(x) = -\sin(x)$, which satisfies $|f''(x)| \leqslant 1$. It follows that for any x, we have

$$|\sin(x) - x| \leqslant \frac{1|x|^2}{2!}.$$

Recall that $|a - b| \leqslant c$ is the same thing as $-c \leqslant a - b \leqslant c$. In particular, for $x = \frac{1}{n}$, we have

$$-\frac{1}{2n^2} \leqslant \sin\left(\frac{1}{n}\right) - \frac{1}{n}.$$

We conclude that $\sin\left(\frac{1}{n}\right) \geqslant \frac{1}{n} - \frac{1}{2n^2}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges ($= \infty$) and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. It follows that $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{2}{n^2} = \infty$, so $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges ($= \infty$) as well.

2. Find a bound on $R_1 \arctan(x)$ which is valid for all x with $0 \leqslant x \leqslant 1$ and use this to show that the sum

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2}\right)$$

converges.

Solution: Set $f(x) = \arctan(x)$ and recall that $T_1 \arctan(x) = x$. To find a bound on $R_1 \arctan(x)$, we need to compute $f''(x)$. Recall that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$, so that $f''(x) = \frac{-2x}{(1+x^2)^2}$. Now, we notice that for x satisfying $0 \leqslant x \leqslant 1$,

$$\left|\frac{-2x}{(1+x^2)^2}\right| = \frac{2|x|}{(1+x^2)^2} \leqslant \frac{2}{(1+x^2)^2} \leqslant 2.$$

The first inequality follows because $2|x| \leqslant 2$ for x with $0 \leqslant x \leqslant 1$. The second follows because $(1 + x^2)^2 \geqslant 1$. From this, we see that for all x with $0 \leqslant x \leqslant 1$,

$$|\arctan(x) - T_1 \arctan(x)| \leqslant \frac{2x^2}{2!} = x^2.$$

Recall that $|a-b| \leqslant c$ is the same thing as $-c \leqslant a-b \leqslant c$. In particular, for $x = \frac{1}{n^2}$, we have

$$\arctan\left(\frac{1}{n^2}\right) - \frac{1}{n^2} \leqslant \frac{1}{n^4}$$

It follows that $0 \leqslant \arctan\left(\frac{1}{n^2}\right) \leqslant \frac{1}{n^2} + \frac{1}{n^4}$. From this, we see that $\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2}\right)$ converges.

3. For which values of x does the series $\sum_{n=1}^{\infty} \frac{x^n n!}{(n!)^2} e^{nx}$ converge? Justify your answer.

Solution: First, notice that $\frac{n!}{(n!)^2} = \frac{1}{n!}$. Set $a_n = \frac{x^n}{n!} e^{nx}$. The series converges when $x = 0$ because the terms are all zero. For $x \neq 0$, we apply the ratio test. Notice that

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1} e^{(n+1)x} n!}{(n+1)!|x|^n e^{nx}} = \frac{|x|e^x}{n+1}$$

which tends to zero as $n \to \infty$ for any x. Therefore, the series converges for all x.

4. Find a bound on $R_n(\sin(x) + \cos(x))$ and use this to show that $T_n(\sin(x) + \cos(x))$ converges to $\sin(x) + \cos(x)$ as $n \to \infty$.

Solution: We begin by recalling the Lagrange form of the Taylor remainder term. If $f(x) = \sin(x) + \cos(x)$ then

$$R_n^0 \sin(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

Notice that $f^{(n+1)}(x)$ is of the form $\pm\sin(x) \pm \cos(x)$, so that for all x we know that $|f^{(n+1)}(x)| \leqslant 2$. It follows that for any x,

$$\begin{aligned} |R_n^0 \sin(x) + \cos(x)| &\leqslant \frac{2}{(n+1)!}|x|^{n+1} \\ &= \frac{2|x|^{n+1}}{(n+1)!} \end{aligned}$$

since $\frac{2|x|^{n+1}}{(n+1)!} \to 0$ for any x, this shows that $R_n^0(\sin(x) + \cos(x)) \to 0$ and therefore that $T_n^0(\sin(x) + \cos(x)) \to \sin(x) + \cos(x)$.

5. Find a bound on $R_n e^{2x}$ and use this to show that for every x, $T_n e^{2x}$ converges to e^{2x} as $n \to \infty$.

Solution: We begin by recalling the Lagrange form of the Taylor remainder term. If $f(x) = e^{2x}$ then $f^{(n)}(x) = 2^n e^{2x}$. We see that $|f^{(n+1)}(t)| \leqslant 2^{n+1} e^{2|x|}$ for any t between 0 and x. Then

$$|R_n e^{2x}| \leqslant \frac{2^{n+1} e^{2|x|}}{(n+1)!}|x|^{n+1} \qquad = \frac{e^{2|x|}}{(n+1)!}|2x|^{n+1}.$$

Notice that $e^{2|x|}$ is a number that does not depend on n, so it suffices to observe that $\lim_{n\to\infty} \frac{|2x|^{n+1}}{(n+1)!} = 0$ for all real numbers x. Therefore $\lim_{n\to\infty} R_n e^{2x} = 0$ and so $\lim_{n\to\infty} T_n e^{2x} = e^{2x}$.

6. For which values of x is it true that $T_n \frac{1}{2-x}$ converges to $\frac{1}{2-x}$ as $n \to \infty$? Justify your answer.

   Solution: First, notice that the result cannot hold when $x = 2$ because the function $\frac{1}{2-x}$ is not defined there. For $x \neq 2$, we write

$$\begin{aligned}\frac{1}{2-x} &= \frac{1}{2}\frac{1}{1-\frac{x}{2}},\\ T_n \frac{1}{2-x} &= \frac{1}{2}\sum_{k=0}^{n}\left(\frac{x}{2}\right)^k\\ &= \frac{1}{2}\frac{1-\left(\frac{x}{2}\right)^{n+1}}{1-\frac{x}{2}}\\ &= \frac{1-\left(\frac{x}{2}\right)^{n+1}}{2-x}\end{aligned}$$

   The second to last equality comes from the geometric series formula. Therefore, for $x \neq 2$,

$$\frac{1}{2-x} - T_n \frac{1}{2-x} = \frac{\left(\frac{x}{2}\right)^{n+1}}{2-x}$$

   It follows that $T_n \frac{1}{2-x}$ converges to $\frac{1}{2-x}$ exactly when $|x| < 2$.