MATH 222 (Lectures 1,2,4) **Worksheet 15** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Let $\vec{\mathbf{a}} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$, $\vec{\mathbf{b}} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ and $\vec{\mathbf{c}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Which of the following expressions are nonsense? Evaluate the sensible ones.

   (a) $3\vec{\mathbf{a}} + \vec{\mathbf{b}}$

   (b) $\vec{\mathbf{a}} + \vec{\mathbf{c}}$

   (c) $\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$

   (d) $\vec{\mathbf{a}} - 2\vec{\mathbf{b}}$

   (e) $t\vec{\mathbf{a}}$ where t is a real number.

   (f) $\vec{\mathbf{a}}\vec{\mathbf{b}}$

   (g) $\vec{\mathbf{a}} + 5$

   Solution:

   (a) $3\vec{\mathbf{a}} + \vec{\mathbf{b}} = \begin{pmatrix} 5 \\ 5 \\ 14 \end{pmatrix}$

   (b) nonsense

   (c) nonsense

   (d) $\vec{\mathbf{a}} - 2\vec{\mathbf{b}} = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix}$

   (e) $t\vec{\mathbf{a}} = \begin{pmatrix} 2t \\ t \\ 5t \end{pmatrix}$

   (f) nonsense

   (g) nonsense

2. Let $\vec{\mathbf{a}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Find s and t so that $\begin{pmatrix} 3 \\ 5 \end{pmatrix} = s\vec{\mathbf{a}} + t\vec{\mathbf{b}}$.

   Solution: We can rewrite this problem as

$$\begin{pmatrix} 2s - t \\ 4s + t \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

   We can solve the equation $2s - t = 3$ for t to find that $t = 2s - 3$. Plugging this in to the equation $4s + t = 5$ we find that $4s + 2s - 3 = 5$ so $6s = 8$ and $s = \frac{4}{3}$. Plugging this back in, we find that $t = 2(\frac{4}{3}) - 3 = \frac{-1}{3}$.

3. Let $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find $\vec{\mathbf{a}}^{/\!/}$ and $\vec{\mathbf{a}}^{\perp}$ so that $\vec{\mathbf{a}} = \vec{\mathbf{a}}^{/\!/} + \vec{\mathbf{a}}^{\perp}$, where $\vec{\mathbf{a}}^{/\!/}$ is parallel to $\vec{\mathbf{b}}$ and $\vec{\mathbf{a}}^{\perp}$ is perpendicular to $\vec{\mathbf{b}}$.

Solution:

$$\begin{aligned} \vec{\mathbf{a}}^{/\!/} &= \left(\vec{\mathbf{a}} \cdot \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|}\right) \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} \\ &= \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right) \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|^2} \end{aligned}$$

Observe that $\|\vec{\mathbf{b}}\| = \sqrt{1+1} = \sqrt{2}$ and $\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right) = 1 - 2 = -1$. Then

$$\vec{\mathbf{a}}^{/\!/} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

and

$$\begin{aligned} \vec{\mathbf{a}}^{\perp} &= \vec{\mathbf{a}} - \vec{\mathbf{a}}^{/\!/} \\ &= \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} \\ 2 \\ -\frac{3}{2} \end{pmatrix} \end{aligned}$$

4. Let $\vec{\mathbf{a}} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find $\vec{\mathbf{a}}^{/\!/}$ and $\vec{\mathbf{a}}^{\perp}$ so that $\vec{\mathbf{a}} = \vec{\mathbf{a}}^{/\!/} + \vec{\mathbf{a}}^{\perp}$, where $\vec{\mathbf{a}}^{/\!/}$ is parallel to $\vec{\mathbf{b}}$ and $\vec{\mathbf{a}}^{\perp}$ is perpendicular to $\vec{\mathbf{b}}$.

Solution:

$$\begin{aligned} \vec{\mathbf{a}}^{/\!/} &= \left(\vec{\mathbf{a}} \cdot \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|}\right) \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} \\ &= \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right) \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|^2} \end{aligned}$$

Observe that $\|\vec{\mathbf{b}}\| = \sqrt{1+1} = \sqrt{2}$ and $\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right) = -1 + 2 = 1$. Then

$$\vec{\mathbf{a}}^{/\!/} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

and

$$\begin{aligned}\vec{\mathbf{a}}^{\perp} &= \vec{\mathbf{a}} - \vec{\mathbf{a}}^{/\!/} \\ &= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{2} \\ 2 \\ \frac{3}{2} \end{pmatrix}\end{aligned}$$

5. Find a parametric equation for the line that passes through the points $A = (1, 0, 2)$ and $B = (3, 1, 4)$.

   Solution:

$$\begin{aligned}\vec{AB} &= \begin{pmatrix} 3-1 \\ 1-0 \\ 4-2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\end{aligned}$$

   so a parametric equation for the line is

$$\vec{\mathbf{l}}(t) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} t$$