MATH 222 (Lectures 1,2,4) **Worksheet 2** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Compute $\int x\ln(x)dx$.

   Solution:

$$\begin{aligned}\int x\ln(x)dx &= \int \underbrace{\ln(x)}_{\text{F(x)}}\ \underbrace{xdx}_{\text{G'(x)dx}}\\ &= \underbrace{\frac{x^2}{2}}_{\text{G(x)}}\ \underbrace{\ln(x)}_{\text{F(x)}} - \int \underbrace{\frac{x^2}{2}}_{\text{G(x)}}\ \underbrace{\frac{1}{x}dx}_{\text{F'(x)dx}}\\ &= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int xdx\\ &= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C\end{aligned}$$

2. Compute $\int \arctan(x)dx$.

   Solution:

$$\begin{aligned}\int \arctan(x)dx &= \int \underbrace{\arctan(x)}_{\text{F(x)}}\ \underbrace{dx}_{\text{G'(x)dx}}\\ &= \underbrace{x}_{\text{G(x)}}\ \underbrace{\arctan(x)}_{\text{F(x)}} - \int \underbrace{x}_{\text{G(x)}}\ \underbrace{\frac{1}{1+x^2}dx}_{\text{F'(x)dx}}\\ &= x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C\end{aligned}$$

3. Compute $\int \cos^2(x)\sin^2(x)dx$.

   Solution:

$$\begin{aligned}\int \cos^2(x)\sin^2(x)dx &= \int (\cos(x)\sin(x))^2\,dx\\ &= \int \left(\frac{1}{2}\sin(2x)\right)^2 dx\\ &= \frac{1}{4}\int \sin^2(2x)dx\\ &= \frac{1}{8}\int 1-\cos(4x)dx\\ &= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C\end{aligned}$$

4. Compute $\int \sin^5(x)dx$.

Solution:

$$\begin{aligned}
\int \sin^5(x)dx &= \int \sin^4(x)\sin(x)dx \\
&= \int \left(\sin^2(x)\right)^2 \sin(x)dx \\
&= \int \left(1-\cos^2(x)\right)^2 \sin(x)dx \qquad\qquad u=\cos(x)\ du=-\sin(x)dx \\
&= -\int \left(1-u^2\right)^2 du \\
&= -\int 1-2u^2+u^4du \\
&= -u+\frac{2}{3}u^3-\frac{1}{5}u^5+C \\
&= -\cos(x)+\frac{2}{3}\cos^3(x)-\frac{1}{5}\cos^5(x)+C
\end{aligned}$$

5. (a) Compute

$$\int_0^\pi \cos(x)dx \qquad \text{and} \qquad \int_0^\pi x^2\cos(x)dx.$$

(b) Use the identity

$$\int x^n\cos(x)dx = x^n\sin(x)+nx^{n-1}\cos(x)-n(n-1)\int x^{n-2}\cos(x)dx$$

and part (a.) to compute $\int_0^\pi x^4\cos(x)dx$.

(c) Show that

$$\int x^n\cos(x)dx = x^n\sin(x)+nx^{n-1}\cos(x)-n(n-1)\int x^{n-2}\cos(x)dx.$$

Hint: the steps are very similar to the steps used to compute $\int_0^\pi x^2\cos(x)dx$ in $(a.)$.

Solution:

(a) The first integral is $\int_0^\pi \cos(x)dx = \sin(x)\Big|_{x=0}^{x=\pi} = 0$. For the second, we use integration by parts:

$$\begin{aligned}\int_0^\pi \underbrace{x^2}_{F(x)} \underbrace{\cos(x)dx}_{G'(x)dx} &= \underbrace{x^2}_{F(x)} \underbrace{\sin(x)}_{G(x)}\Big|_0^\pi - \int_0^\pi \underbrace{2x}_{F'(x)} \underbrace{\sin(x)}_{G(x)} dx \\ &= 0 - 2\int_0^\pi \underbrace{x}_{f(x)} \underbrace{\sin(x)dx}_{g'(x)dx} \\ &= -2\left(\underbrace{x}_{f(x)} \cdot \underbrace{(-\cos(x))}_{g(x)}\Big|_0^\pi - \int_0^\pi \underbrace{1}_{f'(x)} \cdot \underbrace{(-\cos(x))}_{g(x)} dx \right) \\ &= -2\pi - 2\int_0^\pi \cos(x)dx = -2\pi\end{aligned}$$

(b) The identity says that

$$\begin{aligned}\int_0^\pi x^4\cos(x)dx &= \underbrace{\left[x^4\sin(x) + 4x^3\cos(x)\right]_{x=0}^{x=\pi}}_{-4\pi^3} - 4\cdot 3\cdot \underbrace{\int_0^\pi x^2\cos(x)dx}_{-2\pi} \\ &= 24\pi - 4\pi^3\end{aligned}$$

(c) As in part (a.), we integrate by parts twice to obtain this result: first we see that

$$\int \underbrace{x^n}_{F(x)} \underbrace{\cos(x)dx}_{G'(x)dx} = \underbrace{x^n}_{F(x)} \cdot \underbrace{\sin(x)}_{G(x)} - \int \underbrace{nx^{n-1}}_{F'(x)} \cdot \underbrace{\sin(x)}_{G(x)} dx$$

so we need to compute $\int x^{n-1}\sin(x)dx$, which we can do with another integration by parts:

$$\begin{aligned}\int \underbrace{x^{n-1}}_{f(x)} \underbrace{\sin(x)dx}_{g'(x)dx} &= \underbrace{x^{n-1}}_{f(x)} \cdot \underbrace{(-\cos(x))}_{g(x)} - \int \underbrace{(n-1)x^{n-2}}_{f'(x)} \cdot \underbrace{(-\cos(x))}_{g'(x)} dx \\ &= -x^{n-1}\cos(x) + (n-1)\int x^{n-2}\cos(x)dx.\end{aligned}$$

Plugging this in, we see that

$$\begin{aligned}\int x^n\cos(x)dx &= x^n\sin(x) - n\int x^{n-1}\sin(x)dx \\ &= x^n\sin(x) - n\left(-x^{n-1}\cos(x) + (n-1)\int x^{n-2}\cos(x)dx\right) \\ &= x^n\sin(x) + nx^{n-1}\cos(x) - n(n-1)\int x^{n-2}\cos(x)dx.\end{aligned}$$