MATH 222 (Lectures 1,2,4) **Worksheet 3** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Compute $\int \frac{1}{x^2-4}dx$.

   Solution: We begin by computing the partial fraction decomposition of $\frac{1}{x^2-4}$.

$$\begin{aligned}\frac{1}{x^2-4} &= \frac{1}{(x-2)(x+2)}\\ &= \frac{A}{x-2}+\frac{B}{x+2}\\ 1 &= A(x+2)+B(x-2)\end{aligned}$$

   Setting $x=2$ gives that $A=\frac{1}{4}$ and setting $x=-2$ gives that $B=\frac{-1}{4}$. Therefore

$$\begin{aligned}\int \frac{1}{x^2-4}dx &= \frac{1}{4}\int \frac{1}{x-2}dx - \frac{1}{4}\int \frac{1}{x+2}dx\\ &= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C\\ &= \frac{1}{4}\ln|\frac{x-2}{x+2}| + C\end{aligned}$$

2. Compute $\int \frac{1}{x^2+6x+10}dx$.

   Solution: We can see that $x^2+6x+10$ has no real roots by noticing that the discriminant in the quadratic formula (b^2-4ac), which is equal to $6^2-4(10)=-4$, is negative. Therefore for this problem, we complete the square.

$$\begin{aligned}\int \frac{1}{x^2+6x+10} &= \int \frac{1}{(x+3)^2+1}dx\\ &= \int \frac{1}{u^2+1}du \qquad\qquad u=x+3 \quad du=dx\\ &= \arctan(u)+C\\ &= \arctan(x+3)+C\end{aligned}$$

3. Compute $\int \frac{x^3}{x^2+2}dx$.

   Solution: Notice that the degree of the numerator is greater than or equal to the degree of the denominator, so we need to do some kind of polynomial division to get this into a form we can work with. We compute

$$\begin{array}{r|l} & \quad\quad x \\ \hline x^2+2 & x^3 \\ & -x^3-2x \\ \hline & \quad -2x \end{array}$$

and therefore

$$\begin{aligned}
\int \frac{x^3}{x^2+2}dx &= \int x - \frac{2x}{x^2+2}dx \\
&= \frac{x^2}{2} - \int \frac{1}{u}du & u = x^2+2 \quad du = 2xdx \\
&= \frac{x^2}{2} - \ln|u| + C \\
&= \frac{x^2}{2} - \ln(x^2+2) + C
\end{aligned}$$

4. Compute $\int \frac{x}{x^2-1}dx$.

Solution:

$$\begin{aligned}
\int \frac{x}{x^2-1} &= \frac{1}{2}\int \frac{1}{u}du & u = x^2-1 \quad \frac{du}{2} = xdx \\
&= \frac{1}{2}\ln|u| + C \\
&= \frac{1}{2}\ln|x^2-1| + C
\end{aligned}$$

5. Compute $\int \frac{dt}{2+e^{2t}}$.

Solution:

$$\int \frac{dt}{2+e^{2t}} = \int \frac{du}{u(1+u^2)} \qquad u = e^t \quad \frac{1}{u}du = dt$$

we now need to compute the partial fraction decomposition for $\frac{1}{u(1+u^2)}$.

$$\begin{aligned}
\frac{1}{u(2+u^2)} &= \frac{A}{u} + \frac{Bu+C}{2+u^2} \\
1 &= A(2+u^2) + (Bu+C)u \\
0u^2 + 0u + 1 &= (A+B)u^2 + Cu + 2A
\end{aligned}$$

Comparing coefficients, we find that

$$\begin{aligned}
u^2 &: (A+B) = 0 \\
u &: C = 0 \\
1 &: 2A = 1
\end{aligned}$$

which implies that $\frac{1}{u(2+u^2)} = \frac{1}{2u} - \frac{1}{2}\frac{u}{1+u^2}$. Therefore

$$\begin{aligned}
\int \frac{du}{u(2+u^2)} &= \frac{1}{2}\int \frac{1}{u}du - \frac{1}{2}\int \frac{u}{2+u^2}du \\
&= \frac{1}{2}\ln|u| - \frac{1}{4}\ln|2+u^2| + C \\
&= \frac{1}{2}\ln(e^t) - \frac{1}{4}\ln(2+e^{2t}) + C \\
&= \frac{1}{2}t - \frac{1}{4}\ln(2+e^{2t}) + C
\end{aligned}$$

6. Find a recursive formula for $\int \tan^n(x)dx$. Use it to compute $\int \tan^4(x)dx$. Hint: to find the recursive formula you do not need integration by parts but you will need a trigonometric identity.

Solution:

$$\begin{aligned}
\int \tan^n(x)dx &= \int \tan^{n-2}(x)dx \tan^2(x)dx \\
&= \int \tan^{n-2}(x)dx(1 - \sec^2(x))dx \\
&= \int \tan^{n-2}(x)dx - \int \tan^{n-2}\sec^2(x)dx
\end{aligned}$$

Now let $u = \tan(x)$. Then

$$\begin{aligned}
\int \tan^{n-2}\sec^2(x)dx &= \int u^{n-2}du \\
&= \frac{1}{n-1}u^{n-1} \\
&= \frac{1}{n-1}\tan^{n-1}(x)
\end{aligned}$$

Putting this all together,

$$\int \tan^n(x)dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)dx$$

Using our formula,

$$\begin{aligned}
\int \tan^4(x)dx &= \frac{1}{4-1}\tan^{4-1}(x) - \int \tan^{4-2}(x)dx \\
&= \frac{1}{3}\tan^3(x) - \int \tan^2(x)dx \\
&= \frac{1}{3}\tan^3(x) - \left(\frac{1}{2-1}\tan^{2-1}(x) - \int \tan^{2-2}(x)dx\right) \\
&= \frac{1}{3}\tan^3(x) - \left(\tan(x) - \int 1dx\right) \\
&= \frac{1}{3}\tan^3(x) - (\tan(x) - x) + c \\
&= \frac{1}{3}\tan^3(x) - \tan(x) + x + c
\end{aligned}$$