MATH 222 (Lectures 1,2,4) **Worksheet 4Solutions**

Please inform your TA if you find any errors in the solutions.

1. Compute $\int \sqrt{16-x^2}dx$.

   Solution:

$$\begin{aligned}
\int \sqrt{16-x^2}dx &= \int \left(\sqrt{16-16\sin^2(\theta)}\right)4\cos(\theta)d\theta && x = 4\sin(\theta)\\
&= \int 4\cos(\theta)\sqrt{16(1-\sin^2(\theta))}d\theta && dx = 4\cos(\theta)d\theta\\
&= 16\int \cos^2(\theta)d\theta\\
&= 16\int \frac{1+\cos(2\theta)}{2}d\theta\\
&= 16\left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right) + C\\
&= 8\ \text{arcsin}(\frac{x}{4}) + 4\sin(2\text{arcsin}(\frac{x}{4})) + C\\
&= 8\ \text{arcsin}(\frac{x}{4}) + 8\sin(\text{arcsin}(\frac{x}{4}))\cos(\text{arcsin}(\frac{x}{4})) + C\\
&= 8\ \text{arcsin}(\frac{x}{4}) + 2x\sqrt{1-\left(\frac{x}{4}\right)^2} + C
\end{aligned}$$

2. Compute $\int t\sqrt{1-t^2}$dt.

   Solution:

$$\begin{aligned}
\int t\sqrt{1-t^2}dt &= \frac{-1}{2}\int \sqrt{u}du && u = 1-t^2 && \frac{-1}{2}du = tdt\\
&= \frac{-1}{3}u^{\frac{3}{2}} + C\\
&= \frac{-1}{3}\left(1-t^2\right)^{\frac{3}{2}} + C
\end{aligned}$$

3. Compute $\int \frac{x^3}{\sqrt{4-x^2}}dx$.

Solution:

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^2}}dx &= \int \frac{8\sin^3(\theta)}{\sqrt{4-4\sin^2(\theta)}} 2\cos(\theta)d\theta & x &= 2\sin(\theta) \\
&= 8\int \sin^3(\theta)d\theta & dx &= 2\cos(\theta)d\theta \\
&= 8\int (1-\cos^2(\theta))\sin(\theta)d\theta & u &= \cos(\theta) \\
&= -8\int (1-u^2)du & du &= -\sin(\theta)d\theta \\
&= \frac{8}{3}u^3 - 8u + C \\
&= \frac{8}{3}\cos^3(\theta) - 8\cos(\theta) + C \\
&= \frac{8}{3}\cos^3\left(\arcsin\left(\frac{x}{2}\right)\right) - 8\cos\left(\arcsin\left(\frac{x}{2}\right)\right) + C \\
&= \frac{8}{3}\left(1-\left(\frac{x}{2}\right)^2\right)^{\frac{3}{2}} - 8\sqrt{1-\left(\frac{x}{2}\right)^2} + C
\end{aligned}$$

4. Compute $\int e^{4x}\sqrt{1-e^{2x}}dx$.

Solution:

$$\begin{aligned}
\int e^{4x}\sqrt{1-e^{2x}}dx &= \int u^3\sqrt{1-u^2}du && u = e^x \\
&= \int \sin^3(\theta)\sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta && u = \sin(\theta) \\
&= \int \sin^3(\theta)\cos^2(\theta)d\theta \\
&= \int \sin^2(\theta)\cos^2(\theta)\sin(\theta)d\theta \\
&= \int \left(1-\cos^2(\theta)\right)\cos^2(\theta)\sin(\theta)d\theta && v = \cos(\theta) \\
&= -\int \left(1-v^2\right)v^2dv \\
&= \int v^4 - v^2dv \\
&= \frac{1}{5}v^5 - \frac{1}{3}v^3 + C \\
&= \frac{1}{5}\cos^5(\theta) - \frac{1}{3}\cos^3(\theta) + C \\
&= \frac{1}{5}\cos^5(\arcsin(e^x)) - \frac{1}{3}\cos^3(\arcsin(e^x)) + C \\
&= \frac{1}{5}\left(1-e^{2x}\right)^{\frac{5}{2}} - \frac{1}{3}\left(1-e^{2x}\right)^{\frac{3}{2}} + C
\end{aligned}$$

5. Compute $\int \frac{x}{x^2-2x+1}dx$.

Solution: First, notice that $x^2-2x+1=(x-1)^2$. We can use partial fractions to compute this integral:

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$
$$x = A(x-1) + B$$

Now, plugging in $x=1$ gives that $B=1$ and plugging in $x=0$ (or anything) lets us solve for $A=1$. Alternatively, we can rewrite the previous line as

$$x + 0 = Ax + (B-A)$$

and equating coefficients of x and 1 to obtain that $A=1$ and $B-A=0$. We now compute

$$\begin{aligned}\int \frac{x}{x^2-2x+1}dx &= \int \frac{1}{x-1}dx + \int \frac{1}{(x-1)^2}dx \\ &= \int \frac{1}{u}du + \int \frac{1}{u^2}du \qquad\qquad u = x-1 \quad du = dx \\ &= \ln|u| - \frac{1}{u} + C \\ &= \ln|x-1| - \frac{1}{x-1} + C\end{aligned}$$

6. Compute $\int_0^1 \ln(2t+1)dt$.

Solution:

$$\begin{aligned}\int_0^1 \ln(2t+1)dt &= \frac{1}{2}\int_{t=0}^{t=1} \underbrace{\ln(u)}_{\text{F(u)}} \underbrace{du}_{\text{G'(u)du}} \qquad\qquad u = 2t+1 \quad \frac{1}{2}du = dt \\ &= \underbrace{u}_{\text{G(u)}} \underbrace{\ln(u)}_{\text{F(u)}} |_{t=0}^{t=1} - \int_{t=0}^{t=1} \underbrace{u}_{\text{G(u)}} \underbrace{\frac{1}{u}du}_{\text{F'(u)du}}\end{aligned}$$