MATH 222 (Lectures 1,2,4) **Worksheet 5** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Compute $\int \frac{1}{t\ln(t)\sqrt{\ln^2(t)-1}}dt$.

   Solution:

$$\begin{aligned}
\int \frac{1}{t\ln(t)\sqrt{\ln^2(t)-1}}dt &= \int \frac{1}{u\sqrt{u^2-1}}du & u = \ln(t) \quad du = \frac{1}{t}dt \\
&= \int \frac{\sec(\theta)\tan(\theta)}{\sec(\theta)\sqrt{\sec^2(\theta)-1}}d\theta & u = \sec(\theta) \quad du = \sec(\theta)\tan(\theta)d\theta \\
&= \int d\theta \\
&= \theta + C \\
&= \text{arcsec}(u) + C \\
&= \text{arcsec}(\ln(t)) + C
\end{aligned}$$

2. Compute $\int \frac{dx}{\sqrt{1-e^{2x}}}$.

   Solution:

$$\begin{aligned}
\int \frac{dx}{\sqrt{1-e^{2x}}} &= \int \frac{du}{u\sqrt{1-u^2}} & u = e^x \quad \frac{du}{u} = dx \\
&= \int \frac{\cos(\theta)}{\sin(\theta)\sqrt{1-\sin^2(\theta)}} & u = \sin(\theta) \quad du = \cos(\theta)d\theta \\
&= \int \csc(\theta)d\theta \\
&= -\ln|\csc(\theta) + \cot(\theta)| + C \\
&= -\ln|\csc(\arcsin(u)) + \cot(\arcsin(u))| + C \\
&= -\ln|\frac{1}{u} + \frac{\sqrt{1-u^2}}{u}| + C \\
&= \ln|\frac{e^x}{1+\sqrt{1-e^{2x}}}| + C \\
&= x - \ln(1+\sqrt{1-e^{2x}}) + C
\end{aligned}$$

3. Compute $\int \frac{1}{\sqrt{x} + x^{\frac{3}{2}}}dx$.

Solution:

$$\int \frac{1}{\sqrt{x}+x^{\frac{3}{2}}} = \int \frac{2udu}{u(1+u^2)} \qquad u=\sqrt{x} \quad \underbrace{du = \frac{1}{2\sqrt{x}}dx}_{\text{2udu = dx}}$$

$$\begin{aligned} &= 2\int \frac{du}{1+u^2} \\ &= 2\text{arctan}(u) \\ &= 2\text{arctan}(\sqrt{x}) \end{aligned}$$

4. Compute $\int e^{4x}\sqrt{1-e^{2x}}dx$.

   Solution:

$$\begin{aligned} \int e^{4x}\sqrt{1-e^{2x}}dx &= \int u^3\sqrt{1-u^2}du & u = e^x \\ &= \int \sin^3(\theta)\sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta & u=\sin(\theta) \\ &= \int \sin^3(\theta)\cos^2(\theta)d\theta \\ &= \int \sin^2(\theta)\cos^2(\theta)\sin(\theta)d\theta \\ &= \int \left(1-\cos^2(\theta)\right)\cos^2(\theta)\sin(\theta)d\theta & v=\cos(\theta) \\ &= -\int \left(1-v^2\right)v^2dv \\ &= \int v^4 - v^2dv \\ &= \frac{1}{5}v^5 - \frac{1}{3}v^3 + C \\ &= \frac{1}{5}\cos^5(\theta) - \frac{1}{3}\cos^3(\theta) + C \\ &= \frac{1}{5}\cos^5(\text{ arcsin}(e^x)) - \frac{1}{3}\cos^3(\text{ arcsin}(e^x)) + C \\ &= \frac{1}{5}\left(1-e^{2x}\right)^{\frac{5}{2}} - \frac{1}{3}\left(1-e^{2x}\right)^{\frac{3}{2}} + C \end{aligned}$$

5. Compute $\int \frac{1}{(1+x^2)\sqrt{1+\text{arctan}^2(x)}}dx$.

Solution:

$$\begin{aligned}
&\int \frac{1}{(1+x^2)\sqrt{\arctan^2(x)+1}}dx \\
&= \int \frac{1}{\sqrt{u^2+1}}du && u=\arctan(x) \quad du = \frac{1}{1+x^2}dx \\
&= \int \frac{\sec^2(\theta)}{\sqrt{\tan^2(\theta)+1}}d\theta && u=\tan(\theta) \quad du=\sec^2(\theta)d\theta \\
&= \int \sec(\theta)d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)|+C \\
&= \ln|\sqrt{u^2+1}+u|+C && \arctan(u)=\theta \\
&= \ln|\sqrt{\arctan^2(x)+1}+\arctan(x)|+C
\end{aligned}$$

6. Compute $\int_4^\infty \frac{1}{x^3-x}dx$

Solution: Use partial fractions to write

$$\frac{1}{x^3-x} = \frac{1}{2(x+1)} + \frac{1}{2(x-1)} - \frac{1}{x}$$

so that

$$\begin{aligned}
\int \frac{1}{x^3-x}dx &= \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| - \ln|x| + C \\
&= \ln\left(\frac{\sqrt{|x^2-1|}}{|x|}\right) + C.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_4^\infty \frac{1}{x^3-x}dx &= \lim_{a\to\infty}\int_4^a \frac{1}{x^3-x}dx \\
&= \lim_{a\to\infty}\left[\ln\left(\frac{\sqrt{|x^2-1|}}{|x|}\right)\right]_4^a \\
&= \lim_{a\to\infty}\ln\left(\frac{\sqrt{|a^2-1|}}{|a|}\right) - \ln\left(\frac{\sqrt{15}}{4}\right) \\
&= -\ln\left(\frac{\sqrt{15}}{4}\right)
\end{aligned}$$

7. Compute $\int_7^\infty \frac{x+1}{(x^2+1)(x-1)}dx$

Solution: Use partial fractions to write

$$\frac{x+1}{(x^2+1)(x-1)} = \frac{1}{x-1} - \frac{x}{x^2+1}$$

so that

$$\begin{aligned}
\int \frac{x+1}{(x^2+1)(x-1)}dx &= \int \frac{1}{x-1} - \frac{x}{x^2+1}dx \\
&= \ln|x-1| - \frac{1}{2}\ln|x^2+1| + C \\
&= \ln\left(\frac{|x-1|}{\sqrt{x^2+1}}\right) + C.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_7^\infty \frac{x+1}{(x^2+1)(x-1)}dx &= \lim_{b\to\infty} \int_7^b \frac{x+1}{(x^2+1)(x-1)}dx \\
&= \lim_{b\to\infty} \left[\ln\left(\frac{|x-1|}{\sqrt{x^2+1}}\right)\right]_7^b \\
&= \lim_{b\to\infty} \ln\left(\frac{|b-1|}{\sqrt{b^2+1}}\right) - \ln\left(\frac{|7-1|}{\sqrt{7^2+1}}\right) \\
&= \ln\left(\frac{\sqrt{50}}{6}\right)
\end{aligned}$$