MATH 222 (Lectures 1,2,4) **Worksheet 8** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Find the general solution to the differential equation

$$\frac{1}{2x}\frac{dy}{dx} = y + e^{x^2}$$

Solution: We begin by writing the problem in standard form as

$$\frac{dy}{dx} - 2xy = 2xe^{x^2}$$

The integrating factor for this problem is $m(x) = e^{\int -2x dx} = e^{-x^2}$. If we multiply through by e^{-x^2}, then the equation becomes separable and we can find the general solution directly directly.

$$\begin{aligned} e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y &= 2x \\ \frac{d(e^{-x^2}y)}{dx} &= 2x \\ \int d(e^{-x^2}y) &= \int 2x dx \\ e^{-x^2}y &= x^2 + C \\ y(x) &= x^2e^{x^2} + Ce^{x^2} \end{aligned}$$

2. Find a particular solution to the differential equation

$$\begin{aligned} \frac{1+x^3}{3x^2}\frac{dy}{dx} &= 1 - y(x) \\ y(1) &= 2 \end{aligned}$$

Solution: We first put the equation into standard form

$$\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{3x^2}{1+x^3}$$

The integrating factor for this problem is $m(x) = (1 + x^3)$ and the solution is

$$\begin{aligned} y(x) &= \frac{1}{1+x^3}\left(\int 3x^2 dx\right) \\ &= \frac{1}{1+x^3}\left(x^3 + C\right) \end{aligned}$$

The initial condition $y(1) = 2$ gives that $\frac{1+C}{2} = 2$, so $C = 3$ and $y(x) = \frac{3+x^3}{1+x^3}$.

3. A tank starts with 100 liters of water and 1,000 bacteria in it. For now we assume the bacteria do not reproduce. Let $B(t)$ be the number of bacteria in the tank as a function of time, where t is in hours. For each of the situations below, write down a first order differential equation satisfied by $B(t)$, of the form $B' = f(t, B)$. You **do not** need to solve it.

   (a) A little goblin is pouring bacteria into the tank at a rate of 2015 bacteria per hour.

   (b) Like part (a), but we are also draining the tank at a rate of 3 L/hr.

   (c) Like part (b), but now the bacteria are reproducing. This is a strain of bacteria which, if left alone, will double its population every hour.

   Solution:

   (a) $B' = 2015$

   (b) $B' = 2015 - 3\frac{B}{100-3t}$

   (c) $B' = 2015 - 3\frac{B}{100-3t} + \ln(2)B$

   To get the last part we need to know the exponential growth rate of the bacteria. We know that if left alone, the population obeys $P(t) = P_0 2^t$, and so satisfies the differential equation $P' = \ln(2)P$. Thus the exponential growth rate is $ln(2)$.

4. Find an exact solution to the following initial value problem, then use Euler's method with step size $\Delta x = .1$ to estimate $y(.2)$

$$\begin{aligned} \frac{dy}{dx} &= 2xy + x \\ y(0) &= 0 \end{aligned}$$

   Solution: We begin by putting the differential equation into standard form

$$\frac{dy}{dx} - 2xy = x$$

   The integrating factor for this problem is $m(x) = e^{\int -2x dx} = e^{-x^2}$. Multiplication turns this into

$$\underbrace{e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y}_{\frac{d}{dx}e^{-x^2}y} = xe^{-x^2}$$

$$\begin{aligned} e^{-x^2}y &= \int xe^{-x^2}dx \\ &= \frac{-1}{2}e^{-x^2} + C \\ y(x) &= -\frac{1}{2} + Ce^{x^2} \end{aligned}$$

   Substituting in the initial condition $y(0) = 0$ gives that $y(x) = \frac{1}{2}e^{x^2} - \frac{1}{2}$.

To approximate $y(.2)$ we first need an approximation for $y(.1)$.

$$y(.1) \approx y(0) + \frac{dy}{dx}(0)\Delta x$$

where $\frac{dy}{dx}(0) = 2(0)y(0) + 0 = 0$. So we have $\frac{dy}{dx}(0) \approx y(0) + 0(.1) = 0$. We now have

$$y(.2) \approx y(.1) + \frac{dy}{dx}(.1)\Delta x$$

where $\frac{dy}{dx}(.1) = 2(.1)y(.1) + .1 = 2(.1)(0) + .1 = .1$. Then, we have

$$\begin{aligned} y(.2) &\approx y(.1) + \frac{dy}{dx}(.1)\Delta x \\ &\approx 0 + .1(.1) \\ &= .01 \end{aligned}$$

5. $y(x)$ is a function satisfying $y'' = y' + y + x$, $y(0) = 1$ and $y'(0) = 2$. Approximate $y(0.2)$ using Euler's method with step size 0.1. You will need to modify the version of Euler's method we learned in class to handle a second-order equation like this one. I got $y(0.2) \approx 1.43$.

   Solution:

$$\begin{aligned} y(0) &= 1 \\ y(0.1) &\approx y(0) + 0.1 * y'(0) \\ &= 1 + 0.1 * 2 \\ &= 1.2 \\ y'(0.1) &\approx y'(0) + 0.1 * y''(0) \\ &= 2 + 0.1 * (2 + 1 + 0) \\ &= 2.3 \\ y(0.2) &\approx y(0.1) + 0.1 * y'(0.1) \\ &\approx 1.2 + 0.1 * 2.3 \\ &= 1.43 \end{aligned}$$