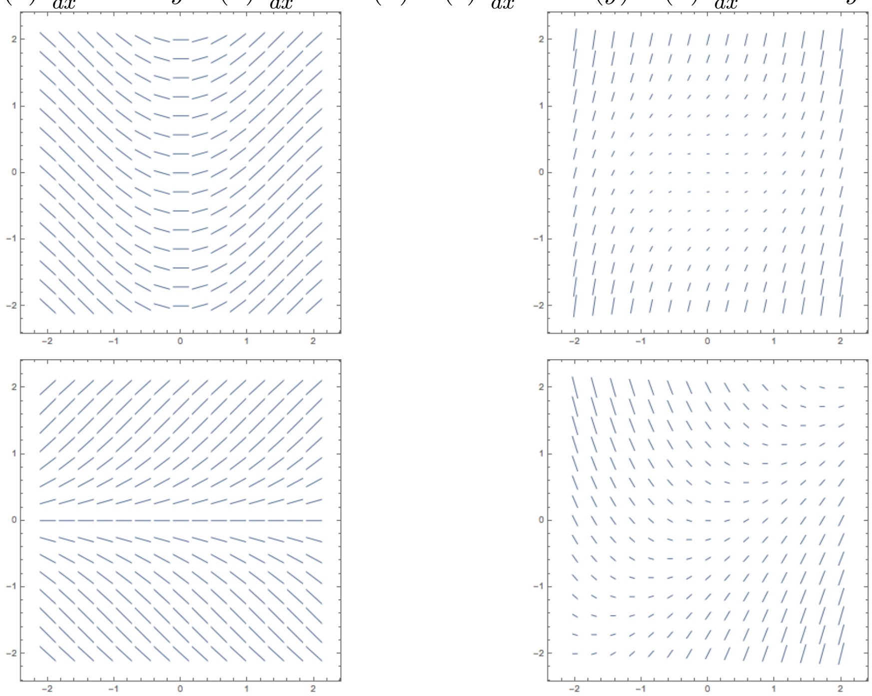
MATH 222 (Lectures 1,2,4) **Worksheet 9** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Identify which of the following differential equations are associated to each of the following direction fields:

(a) $\frac{dy}{dx} = x - y$ (b) $\frac{dy}{dx} = \sin(x)$ (c) $\frac{dy}{dx} = \sin(y)$ (d) $\frac{dy}{dx} = x^2 + y^2$



Solution:

$(b)\frac{dy}{dx} = \sin(x)$ (d) $\frac{dy}{dx} = x^2 + y^2$

$(c)\frac{dy}{dx} = \sin(y)$ (a) $\frac{dy}{dx} = x - y$

2. Solve the following initial value problem exactly, then compute its degree two Taylor polynomial around zero and use this to compute an estimate for $y(.3)$. Then use Euler's method with step size $\Delta x = .1$ to estimate $y(.3)$.

$$\frac{dy}{dx} = -2xy$$
$$y(0) = 1$$

Solution: This differential equation is separable and the solution is $y(x) = e^{-x^2}$. To compute its Taylor polynomial, we compute

$$y(x) = e^{-x^2} \qquad y(0) = 1$$
$$y'(x) = -2xe^{-x^2} \qquad y'(0) = 0$$
$$y''(x) = -2e^{-x^2} + 4x^2e^{-x^2} \qquad y''(0) = -2$$

so the degree two Taylor polynomial around zero is $1 + 0 \cdot x - \frac{2}{2!}x^2 = 1 - x^2$. Our first approximation is then $y(.3) \approx 1 - (.3)^2 = .91$.

To use Euler's method with step size .1, we will iteratively compute estimates to $y(.1), y(.2)$ and $y(.3)$.

First, we need an estimate for $y(.1) \approx y(0) + \frac{dy}{dx}(0)\Delta x$, so we need to compute $\frac{dy}{dx}(0)$.

$$\frac{dy}{dx}(0) = -2(0)y(0) = 0$$

so we estimate that

$$\begin{aligned} y(.1) &\approx y(0) + \frac{dy}{dx}(0)(.1) \\ &= 1 + (0)(.5) = 1 \end{aligned}$$

Now we repeat the procedure above to find an approximation for $y(.2)$

$$y(.2) \approx y(.1) + \frac{dy}{dx}(.1)\Delta x$$

To compute $\frac{dy}{dx}(.1)$, we recall that we have the differential equation $\frac{dy}{dx} = -2xy$, so $\frac{dy}{dx}(.1) = -2(.1)y(.5)$. We have estimated that $y(.1) = 1$, so we use that in our computation, and get $\frac{dy}{dx}(.5) \approx -2(.1)(1) = -.2$. This gives us the approximation

$$\begin{aligned} y(.2) &\approx y(.1) + \frac{dy}{dx}(.1)\Delta x \\ &\approx 1 + (-.2)(.1) = .98 \end{aligned}$$

Finally, we compute an approximation to $y(.3) \approx y(.2) + \frac{dy}{dx}(.2)\Delta x$. We first compute

$$\begin{aligned} \frac{dy}{dx}(.2) &= -2(.2)y(.2) \\ &\approx -2(.2)(.98) \\ &= -.392 \end{aligned}$$

Then our approximation to $y(.3)$ is

$$\begin{aligned} y(.3) &\approx y(.2) + \frac{dy}{dx}(.2)\Delta x \\ &\approx .98 + (-.392)(.1) \\ &= .9408 \end{aligned}$$

As a comment, the true value of $e^{-(.3)^2}$ is about .914.

3. Compute the degree two Taylor polynomial of the function $f(x) = e^{\tan(x)}$ around 0. Use this to estimate $e^{\tan(.1)}$.

Solution:

$$\begin{aligned} f(x) &= e^{\tan(x)} \\ f'(x) &= \sec^2(x)e^{\tan(x)} \\ f^{(2)}(x) &= 2\sec^2(x)\tan(x)e^{\tan(x)} + \sec^4(x)e^{\tan(x)} \end{aligned}$$

We evaluate these at zero.

$$\begin{aligned} f(0) &= e^{\tan(0)} = 1 \\ f'(0) &= \sec^2(0)e^{\tan(0)} = 1 \\ f^{(2)}(0) &= 2\sec^2(0)\tan(0)e^{\tan(0)} + \sec^4(0)e^{\tan(0)} = 1 \end{aligned}$$

This gives the Taylor polynomial $1 + \frac{1}{1!}x + \frac{1}{2!}x^2 = 1 + x + \frac{1}{2}x^2$. We can then approximate $e^{\tan(.1)} \approx 1 + .1 + \frac{1}{2}(.1)^2 = 1.105$.

4. For $f(x) = \int_0^x e^{-t^2}dt$, find the degree three Taylor polynomial of $f(x)$ around 0 and use this to estimate $f(.1)$.

Solution:

$$\begin{aligned} f(x) &= \int_0^x e^{-t^2}dt \\ f'(x) &= e^{-x^2} \\ f^{(2)}(x) &= -2xe^{-x^2} \\ f^{(3)}(x) &= -2e^{-x^2} + 4x^2e^{-x^2} \end{aligned}$$

so that

$$\begin{aligned} f(0) &= \int_0^0 e^{-t^2}dt = 0 \\ f'(0) &= e^{-0^2} = 1 \\ f^{(2)}(0) &= -2(0)e^{-0^2} = 0 \\ f^{(3)}(0) &= -2 \end{aligned}$$

so that the degree three Taylor polynomial for $f(x)$ is $x - \frac{2}{3!}x^3 = x - \frac{1}{3}x^3$. Our estimate for $f(.1)$ is therefore $\frac{1}{10} - \frac{1}{3}\frac{1}{10^3} = .099\bar{6}$.

5. A 100 litre vat of water begins with an algae concentration of 1,000 organisms per litre. Suppose that the algae naturally reproduce at a rate of five percent per minute and die at a rate of four percent per minute. If the vat is being drained at a rate of one litre

per minute, what will the algae concentration be ten minutes from now? You should assume that the algae are uniformly distributed in the vat. Remember to define your variables with units.

Solution: We will model the total population of algae in the vat $P(t)$ and then notice that the concentration at time t is given by $\frac{P(t)}{V(t)}$, where $V(t)$ is the volume of water in the vat. The differential equation for the algae population is

$$\begin{aligned} \frac{dP}{dt} &= (.05)P(t) - (.04)P(t) - \frac{P(t)}{V(t)} \\ P(0) &= 1,000(100) = 100,000 \end{aligned}$$

Since $V(t) = 100 - t$, this differential equation becomes

$$\begin{aligned} \frac{dP}{dt} &= \left(.01 - \frac{1}{100-t}\right) P(t) \\ P(0) &= 100,000 \end{aligned}$$

This equation is separable. Recalling that for $t < 100$ we have $\ln(100 - t) = \ln|100 - t|$, it follows that

$$\begin{aligned} \frac{dP}{P} &= \left(.01 - \frac{1}{100-t}\right) dt \\ \ln|P| &= .01t + \ln|100 - t| + C \\ P(t) &= Ae^{.01t+\ln(100-t)} \\ &= A(100 - t)e^{.01t} \end{aligned}$$

The initial condition $P(0) = 100,000$ gives us

$$100,000 = 100A$$

so $A = 1000$ and $P(t) = 1000(100 - t)e.01t$. The concentration in organisms per litre ten minutes from now will be

$$\frac{P(10)}{V(10)} = \frac{1000(100-10)e^{(.01)(10)}}{100-10} = 1000e^{.1}$$